1. [6+6 points] Suppose $X$ is uniformly distributed on $[0,2]$.
   
   (a) Find $E(X^2)$.
   
   Solution:
   
   
   \[
   E(X^2) = \int_0^2 \frac{1}{2} u^2 \, du = \left[ \frac{u^3}{6} \right]_0^2 = \frac{4}{3}.
   \]

   (b) Find $F_X(1)$.
   
   Solution:
   
   \[
   F_X(1) = P(X \leq 1) = \int_0^1 \frac{1}{2} \, du = \frac{1}{2}.
   \]

2. [10+10+8 points] Three random variables are defined: $X$ is exponentially distributed with parameter $\lambda = 2$, $Y$ is Gaussian with $N(2,9)$ and $U$ is uniformly distributed in $[0,1]$. Answer the following:

   (a) Find a function $g$ such that $g(X) = U$. Assume that $g$ is non-decreasing and $g^{-1}$ exists. Show steps for full credit.
   
   Solution: Let $c$ be the value taken by $U$.
   
   
   \[
   F_U(c) = P(U \leq c) = c = P(g(X) \leq c) = P(X \leq g^{-1}(c)) = F_X(g^{-1}(c))
   \]
   
   \[\implies F_X(g^{-1}(c)) = c \implies g^{-1}(c) = F_X^{-1}(c) \implies g(c) = F_X(c)\]

   Since $X \sim \text{Exp}(2)$, hence $g(X) = 1 - e^{-2X} = U$.

   (b) Find $h$ such that $h(U) = Y$. Assume that $h$ is non-decreasing and $h^{-1}$ exists. Show steps for full credit.
   
   Solution: Let $u$ be the value taken by $U$ and $c$ be the value taken by $Y$.
   
   \[
   F_Y(c) = \Phi\left(\frac{c - 2}{3}\right) = P(Y \leq c) = P(h(U) \leq c) = P(U \leq h^{-1}(c)) = F_U(h^{-1}(c)) = h^{-1}(c)
   \]
   
   \[\implies h^{-1}(c) = u = \Phi\left(\frac{c - 2}{3}\right) \implies h(u) = 3\Phi^{-1}(u) + 2\]

   Hence, $h(U) = 3\Phi^{-1}(U) + 2 = Y$.

   (c) Find a function $k$ such that $k(X) = Y$. Assume that $k$ is non-decreasing and $k^{-1}$ exists. Show steps for full credit.
   
   Solution: Let $u$ be the value taken by $X$ and $c$ be the value taken by $Y$.
   
   \[
   F_Y(c) = \Phi\left(\frac{c - 2}{3}\right) = P(Y \leq c) = P(X \leq k^{-1}(c)) = F_X(k^{-1}(c)) = 1 - e^{-2k^{-1}(c)}
   \]
   
   \[\implies k^{-1}(c) = u = -\frac{1}{2} \ln \left(1 - \Phi\left(\frac{c - 2}{3}\right)\right) \implies k(u) = c = 3\Phi^{-1}(1 - e^{-2u}) + 2\]

   Hence, $k(X) = 3\Phi^{-1}(1 - e^{-2X}) + 2 = Y$. Alternatively, $k(X) = h(g(X))$ but you need parts a) and b) to be correct.
3. [12+10 points] Consider the binary hypothesis problem in which the pdfs of \(X\) under hypotheses \(H_0\) and \(H_1\) are given by

\[
\begin{align*}
&f_0(u) = \begin{cases} 
\frac{1}{2} & \text{if } 0 \leq u \leq 2 \\
0 & \text{otherwise}
\end{cases} \\
&f_1(u) = \begin{cases} 
u & \text{if } 0 \leq u < 1 \\
2-u & \text{if } 1 \leq u \leq 2 \\
0 & \text{otherwise}
\end{cases}
\end{align*}
\]

with priors \(\pi_1 = 2\pi_0\).

(a) Write an expression for the likelihood function \(\Lambda(u)\) and find the MAP rule.

Solution:

\[
\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \begin{cases} 
2u & \text{if } 0 \leq u < 1 \\
4-2u & \text{if } 1 \leq u \leq 2 \\
0 & \text{otherwise}
\end{cases}
\]

Since, the threshold for the MAP rule is \(\frac{\pi_0}{\pi_1} = 0.5\), the likelihood ratio test is given as:

\[
\Lambda(X) \begin{cases} > 0.5 & \text{declare } H_1 \text{ is true} \\
< 0.5 & \text{declare } H_0 \text{ is true}
\end{cases}
\]

Hence, the MAP decision rule is given by:

\[
\begin{align*}
&\frac{1}{4} < X < \frac{7}{4} : \text{ declare } H_1 \text{ is true} \\
&0 < X < \frac{1}{4} \text{ or } \frac{7}{4} < X < 2 : \text{ declare } H_0 \text{ is true}
\end{align*}
\]

(b) Calculate \(p_{\text{miss}}\), \(p_{\text{false-alarm}}\), and error probability \(p_e\).

Solution: Since \(\pi_0 + \pi_1 = 1\) and \(\pi_1 = 2\pi_0 \implies \pi_0 = \frac{1}{3}, \pi_1 = \frac{2}{3}\)

\[
\begin{align*}
p_{\text{miss}} &= P\{\text{declare } H_0|H_1\} = P\{\{0 < X < \frac{1}{4}\} \cup \{\frac{7}{4} < X < 2\}|H_1\} = \frac{1}{16} \\
p_{\text{false-alarm}} &= P\{\text{declare } H_1|H_0\} = P\{\{\frac{1}{4} < X < \frac{7}{4}\}|H_0\} = \frac{3}{4} \\
p_e &= p_{\text{miss}} \times \pi_1 + p_{\text{false-alarm}} \times \pi_0 = \frac{1}{16} \times \frac{2}{3} + \frac{3}{4} \times \frac{1}{3} = \frac{7}{24}
\end{align*}
\]

4. [8+6+8 points] Suppose \(X\) and \(Y\) are discrete-type random variables with the joint pmf given by

\(p_{X,Y}(u, v) = \frac{u+v}{32}\), \(u = 1, 2, v = 1, 2, 3, 4\).

(a) Find \(p_X(u)\), the marginal pmf of \(X\).

Solution: \(p_X(1) = \frac{1+1}{32} + \frac{1+2}{32} + \frac{1+3}{32} + \frac{1+4}{32} = \frac{7}{16}\)
\n\(p_X(2) = \frac{2+1}{32} + \frac{2+2}{32} + \frac{2+3}{32} + \frac{2+4}{32} = \frac{9}{16}\)

Alternatively, \(p_X(u) = \frac{2u+5}{16}, u = 1, 2\).

(b) Find \(P\{Y = 2X\}\)

Solution: \(P\{Y = 2X\} = p_{X,Y}(1, 2) + p_{X,Y}(2, 4) = \frac{3}{32} + \frac{6}{32} = \frac{9}{32}\)
(c) Find $p_{X|Y}(u|2)$, the conditional probability of $X$ given $Y = 2$.

**Solution:**

\[
p_Y(2) = \frac{1+2}{32} + \frac{2+2}{32} = \frac{7}{32}
\]

\[
p_{X|Y}(u|2) = \begin{cases} 
\frac{3}{32}, & u = 1 \\
\frac{7}{32}, & u = 2
\end{cases}
\]

or $p_{X|Y}(u|2) = \frac{u+2}{7}, u = 1, 2$.

5. **[6+10 points]** Suppose $X$ and $Y$ are independent random variables with probability density functions $f_X(u) = \begin{cases} 
3u^2, & 0 \leq u \leq 1 \\
0, & \text{else}
\end{cases}$ and $f_Y(v) = \begin{cases} 
2v, & 0 \leq v \leq 1 \\
0, & \text{else}
\end{cases}$ respectively.

(a) Find the joint pdf of $(X, Y)$.

**Solution:** Since $X$ and $Y$ are independent, the joint pdf is

\[
f_{X,Y}(u,v) = f_X(u)f_Y(v) = \begin{cases} 
6u^2v, & 0 \leq u \leq 1, \ 0 \leq v \leq 1 \\
0, & \text{else}
\end{cases}
\]

(b) Find $P\{X < Y\}$.

**Solution:**

\[
P(X < Y) = \int_0^1 \left( \int_0^v 6u^2vdu \right) dv = \int_0^1 v \left( \int_0^v 6u^2 du \right) dv
\]

\[
= \int_0^1 v \left( 2u^3 \right)_0^v dv = \left. \int_0^1 2v^4 dv = \frac{2}{5}v^5 \right|_0^1 = \frac{2}{5}.
\]

Alternative solution:

\[
P(X < Y) = \int_0^1 \left( \int_u^1 6u^2v dv \right) du = \int_0^1 u^2 \left( \int_u^1 6vdv \right) du
\]

\[
= \int_0^1 u^2 \left( 3v^2 \right)_u^1 du = \int_0^1 (3u^2 - 3u^4) du = \left. \left( u^3 - \frac{3}{5}u^5 \right) \right|_0^1 = \frac{2}{5}.
\]