University of Illinois Spring 2019

ECE 313: Hour Exam I

Wednesday, February 27, 2019
8:45 p.m. — 10:00 p.m.

1. **[6+14 points]** An experiment consists of observing the content of an 8-bit register. We assume that all 256 byte values are equally likely to be observed.

   (a) Let A denote the event that the least significant bit (LSB) is a ZERO. What is $P(A)$?
   Solution: Since the LSB = ZERO, that leaves the top seven bits free to take one of either ONE or ZERO values. There are 2^7 such bytes. Therefore, the $P(A) = \frac{2^7}{2^8} = \frac{1}{2}$.

   (b) Let B denote the event that the register contains 5 ONEs and 3 ZEROes. What is $P(B|A)$?
   Solution: $P(B|A) = \frac{P(AB)}{P(A)}$. To calculate $P(AB)$ note that event AB is the set of outcomes in which the memory word has exactly 3 ZEROes with one ZERO in the LSB. Thus, the 7 non-LSB bits have exactly 5 ONEs and 2 ZEROs. Therefore, $|AB| = \frac{7!}{5!\times 2!} = 21$ and therefore $P(AB) = \frac{21}{256}$ and $P(B|A) = \frac{21}{128}$.

2. **[10+6+8 points]** A 3-faced die is rolled twice. $\omega_1 \in \{1,2,3\}$ and $\omega_2 \in \{1,2,3\}$ are the outcomes of each roll. The random variable X takes values in the set $\{2\omega_1 + \omega_2\}$.

   (a) Determine the set of values u_i that X can take and the probability mass function $P_X(u_i)$.
   Solution: The sample space

   $$\Omega = \{(\omega_1, \omega_2\} = \{(1,1),(1,2),(1,3),(2,1),(2,2),(2,3),(3,1),(3,2),(3,3)\}$$

   and $\{2\omega_1 + \omega_2\} = \{3,4,5,5,6,7,7,8,9\}$ Therefore, $P_X(3) = \frac{1}{9}$, $P_X(4) = \frac{1}{9}$, $P_X(5) = \frac{2}{9}$, $P_X(6) = \frac{1}{9}$, $P_X(7) = \frac{2}{9}$, $P_X(8) = \frac{1}{9}$, $P_X(9) = \frac{1}{9}$.

   (b) Calculate $E[X]$.
   Solution: $E[X] = (3+4+6+8+9)\times\frac{1}{9} + (5+7)\times\frac{2}{9} = \frac{18}{3}$.

   (c) Calculate $E[\frac{1}{2X}]$.
   Solution: $E[\frac{1}{2X}] = \frac{1}{2}E[\frac{1}{X}]$. Using LOTUS, $E[\frac{1}{X}] = (\frac{1}{3}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\frac{1}{9})\times\frac{1}{9} + (\frac{1}{5}+\frac{1}{7})\times\frac{2}{9}$.

3. **[8+8+6 points]** Bob plays a lottery game. He spends \$2 each week to purchase a lottery ticket for this game. Suppose the probability of winning a \$4 prize on each ticket is 1/30, and the probability of winning the Jackpot (the grand prize) on each ticket is 1/300,000,000.

   (a) Bob will purchase a total of 52 tickets in the year 2019 (=52 weeks). What is the expected number of times that Bob wins a \$4 prize this year? Present your answer in an irreducible fraction.
   Solution: Let X denote the number of times that Bob wins a \$4 prize with 52 tickets. X follows a binomial distribution with parameters $n = 52$ and $p = 1/30$.
   $E[X] = np = 52 \times \dfrac{1}{30} = \dfrac{26}{15}$

   (b) Bob has purchased 8 tickets since Jan 1, 2019. What is the probability that Bob has won the \$4 prize twice since Jan 1, 2019? Write down the expression for your answer without calculating the final numerical value.
   Solution: Let Y denote the number of times that Bob wins a \$4 prize with 8 tickets. Y follows a binomial distribution with parameters $n = 8$ and $p = 1/30$.
   $P\{Y = 2\} = \binom{8}{2}(\frac{1}{30})^2(\frac{29}{30})^6$

(c) Find the expected number of tickets needed until a Jackpot is won by Bob (suppose he could play the game forever).

Solution: Let Z denote the number of tickets needed until Bob wins a Jackpot. Z follows a geometric distribution with parameter $p = 1/300,000,000$.
$E[Z] = 1/p = 300,000,000$.

4. **[8+8 points]** Suppose a random variable X is uniformly distributed on $\{2, 4, 6, 8, 10, ..., 2n\}$.

(a) We observe $X = 10$. Find the ML estimate of n.

Solution: Since X has a uniform distribution, $P(X = 10) = \frac{1}{n}$, where $2n \geq 10$. To maximize $P(X = 10)$, we have $n = 5$.

(b) Suppose we make two independent observations of X. Denote the observations by X_1 and X_2. We have $X_1 = 20$ and $X_2 = 16$. Find the ML estimate of n.

Solution: Since X has a uniform distribution, $P(X_1 = 20, X_2 = 16) = \left(\frac{1}{n}\right)^2$, where $2n \geq 20$. To maximize $P(X_1 = 20, X_2 = 16)$, we have $n = 10$.

5. **[12+6 points]** Suppose Z_1, Z_2, Z_3 are i.i.d. Bernoulli random variables with parameter $p = \frac{1}{2}$. The random variable $S = Z_2 Z_3$ if $Z_1 = 1$, and $S = Z_2 + Z_3$ if $Z_1 = 0$.

(a) Find $P(S = 1)$.
Solution:

$$\begin{aligned} P(S=1) &= P(Z_1 = 1, Z_2 = 1, Z_3 = 1) + P(Z_1 = 0, Z_2 = 1, Z_3 = 0) \\ &\quad + P(Z_1 = 0, Z_2 = 0, Z_3 = 1) = \frac{3}{8}. \end{aligned}$$

(b) Find $P(Z_1 = 1 | S = 1)$.
Solution:

$$P(Z_1 = 1|S = 1) = \frac{P(Z_1 = 1, Z_2 = 1, Z_3 = 1)}{P(S = 1)} = \frac{\frac{1}{8}}{\frac{3}{8}} = \frac{1}{3}.$$