University of Illinois Spring 2018

ECE 313: Final Exam

Wednesday, May 9, 2018
1:30 p.m. — 4:30 p.m.

1. **[22 points]** Let X and Y be two jointly continuous random variables with a joint pdf given by:

$$f_{XY}(x,y) = c\,x^2\,y, \text{ for } 0 \le y \le x \le 1,$$

and zero elsewhere. Here, c denotes a constant.

(a) Find the constant c.
Solution: To find the constant, use the fact that

$$1 = \int_x \int_y f_{XY}(x,y)\,dx\,dy = \int_0^1 \int_0^x c\,x^2\,y\,dx\,dy = \int_0^1 \frac{cx^4}{2}\,dx = \frac{c}{10}.$$

This gives $c = 10$.

(b) Find the marginal pdf of X, $f_X(x)$.
Solution: The marginal is obtained via

$$f_X(x) = \int_0^x 10x^2 y\,dy = 5x^4,$$

for $x \in [0,1]$. The pdf is zero otherwise.

(c) Find the probability $P\{Y \le \frac{X}{2}\}$.
Solution: The easiest way to obtain the probability is via integration,

$$P\{Y \le \frac{X}{2}\} = \int_0^1 \int_0^{x/2} 10\,x^2\,y\,dx\,dy = \int_0^1 \frac{5}{4}\,x^4\,dx,$$

which gives the value 1/4.

2. [**16 points**] Consider random variables $X \sim N(\mu, \sigma^2)$, and $Y = 2X + 5$.

(a) Find μ and σ^2 if $Var(Y) = 8$ and $P(Y < 6) = 1/2$.
Solution: $Var(Y) = 4Var(X) = 8$, and therefore $Var(X) = \sigma^2 = 2$.
Since $P(Y < 6) = 1/2$, $E[Y] = 6 = 2E[X] + 5$, and therefore $E[X] = \mu = 0.5$.

(b) For this part, assume $\mu = -0.5$ and $\sigma^2 = 1$. Compute $P(Y^2 - 2Y \le 0)$, and leave your answer as a function of the Q function.
Solution: We have $E[Y] = 2E[X] + 5 = 4$ and $Var(Y) = 4Var(X) = 4$.

$$\begin{aligned} P(Y^2 - 2Y \le 0) &= P(0 \le Y \le 2) \\ &= P(\frac{0-4}{2} \le \frac{Y-4}{2} \le \frac{2-4}{2}) \\ &= P(-2 \le \frac{Y-4}{2} \le -1) \\ &= Q(1) - Q(2) \end{aligned}$$

3. [**24 points**] Given n MPs submitted by ECE students. An MP contains a nasty bug with probability p, independent of other MPs. We run the MPs one by one on our system. The system crashes if an MP contains a nasty bug.

(a) If $n = 3$ and $p = 0.5$, find the probability that the system crashes.
Solution: Let X be the number of MPs that contain a nasty bug. X is a binomial random variable with $p = 0.5$.

$$P(X > 0) = 1 - P(X = 0) = 1 - (0.5)^3 = \frac{7}{8}.$$

(b) If $p = 0.2$, find the probability that the system crashes on the 4th MP.
Solution: Let Y be a geometric random variable with $p = 0.2$.

$$P(Y = 4) = 0.2(1 - 0.2)^3 = \frac{64}{625}.$$

(c) If $p = 0.2$, given the system did not crash after running three MPs, what is the probability that it crashes on the 5th MP?
Solution:

$$P(Y = 5|Y > 3) = (1 - p)p = \frac{4}{25}.$$

(d) If $n = 100$ and $p = 0.01$, find the Poisson approximation of the probability that the system crashes. Leave your answer in terms of powers of e, e.g., ae^b.
Solution:

$$P(X > 0) = 1 - e^{-\lambda} = 1 - e^{-1}.$$

4. **[18 points]** Four fair dice are thrown. Denote by X_i the number showing on the ith die. Let $X = X_1 + X_2$, and $Y = X_3 - X_4$.

(a) Find $Cov(X_1, X_2)$ and $Cov(X + Y, X - Y)$.
Solution:
$Cov(X_1, X_2) = 0$ because X_1 and X_2 are independent.

$$\begin{aligned} Cov(X + Y, X - Y) &= Cov(X, X) - Cov(X, Y) + Cov(X, Y) - Cov(Y, Y) \\ &= Cov(X_1 + X_2, X_1 + X_2) - Cov(X_3 - X_4, X_3 - X_4) \\ &= Var(X_1 + X_2) - Var(X_3 - X_4) \\ &= Var(X_1) + Var(X_2) - Var(X_3) - Var(X_4) \\ &= 0, \end{aligned}$$

since $Var(X_1) = Var(X_2) = Var(X_3) = Var(X_4)$.

(b) Find $\rho_{5X_1+3,2X_3-5}$.
Solution: $\rho_{5X_1+3,2X_3-5} = \rho_{X_1,X_3} = 0$, since X_1 and X_3 are independent.

5. **[18 points]** Let X be a random variable with pdf $f_X(u) = \frac{2u}{a^2} + \frac{2}{a}$ for $u \in (-a, 0)$ and zero otherwise, for some real number a.

(a) For this part of the problem only, assume that $a = 2$ and let $Y = X^2$. Obtain $f_Y(v)$, the pdf of Y, for all v.
Solution: Notice that $X \in (-2, 0)$, so that $Y \in (0, 4)$. Consider the CDF of Y on that set,

$$F_Y(v) = P\{Y \le v\} = P\{X^2 \le v\} = P\{X \ge -\sqrt{v}\} = 1 - F_X(-\sqrt{v}).$$

Taking derivative,

$$f_Y(v) = \frac{d}{dv}F_Y(v) = -f_X(-\sqrt{v})\left(\frac{-1}{2\sqrt{v}}\right) = -\left(\frac{-\sqrt{v}}{2}+1\right)\left(\frac{-1}{2\sqrt{v}}\right) = \frac{1}{2\sqrt{v}} - \frac{1}{4}.$$

Hence, $f_Y(v) = \frac{1}{2\sqrt{v}} - \frac{1}{4}$ for $v \in (0,4)$ and zero else.

(b) For this part of the problem, assume that a is unknown. The experiment is performed, and it is observed that $X = -\frac{1}{3}$. Determine $\hat{a}_{ML}$, the maximum likelihood value of the parameter a.
Solution: The objective is to maximize $f_X\left(-\frac{1}{3}\right)$, the likelihood of observing $X = -\frac{1}{3}$. Taking derivatives,

$$0 = \frac{d}{da}f_X\left(-\frac{1}{3}\right) = \frac{d}{da}\left(\frac{2}{a^2}\left(-\frac{1}{3}\right)+\frac{2}{a}\right) = \frac{4}{3a^3} - \frac{2}{a^2}.$$

Solving for a yields $\hat{a}_{ML} = \frac{2}{3}$.

6. **[12 points]** Let X be a geometric random variable with $p = 0.2$.

(a) Find $E[X]$.
Solution: $E[X] = \frac{1}{0.2} = 5$.

(b) Find $E[X|X > 2]$.
Solution: $E[X|X > 2] = \frac{1}{0.2} + 2 = 7$.

7. **[15 points]** The two parts of the problem are unrelated.

(a) Suppose a fair die is rolled 100 times. What is a rough approximation to the sum of the numbers showing, based on the law of large numbers?
Solution: This is problem 4.10.2 from the text, with the value 1000 changed to 100. Since we have $E[X_i] = \frac{1+2+3+4+5+6+}{6} = 3.5$ for each role outcome variable X_i, $i = 1, \ldots, 100$, the law of large numbers asserts that we expect to see a value close to $100 \cdot 3.5 = 350$.

(b) Suppose each of 1200 real numbers are rounded to the nearest integer and then added. Assume the individual roundoff errors are independent and uniformly distributed over the interval $[-0.5, 0.5]$. The random variable equal to the sum is denoted by S. Using the CLT, find the approximate probability that the absolute value of the sum of the errors is greater than 5.
Solution: This is problem 4.10.7 from your text, with small numerical changes. The uniform random variables $X_i, i = 1, \ldots, 120$ to be summed up have expected value $X_i = 0$ for all i, and $\text{var}(X_i) = \frac{1}{12}$. Hence, the sum of the variables S has expected value $E[S] = 0$ and variance $\text{var}(S) = 120 \cdot \frac{1}{12} = 100$, since the variables are independent. Hence, by the CLT,

$$P\{|S| \geq 5\} = P\left\{\frac{|S|}{10} \geq \frac{5}{10}\right\} = 2Q(0.5).$$

8. **[15 points]** The two parts of the problem are unrelated.

(a) An urn contains 990 blue balls and 10 red balls. Six students each pick a ball independently at random, with replacement, and observe its color. We wish to bound the probability that at least one student picked and observed a red ball. Let S_k denote the event that student k draws a red ball. Note that the probability of interest can be written as $P\{\cup_{k=1}^{6} S_k\}$. Use the union bound when evaluating the probability. Compute the exact value of $P\{\cup_{k=1}^{6} S_k\}$ and compare the results. You may use the fact that $0.99^6 = 0.942$ to aid your comparison.

Solution: By the union bound, $P\{\cup_{k=1}^{6} S_k\} \le \sum_{k=1}^{6} P\{S_k\}$. Since the balls are drawn with replacement, and in each draw we have probability $10/1000 = 0.01$ to draw a red ball, the desired bound equals 0.06. The correct result may be inferred by noting that the probability we seek equals $1-p$, where p is the probability that no red ball is picked. Consequently, $p = 0.99^6 = 0.942$, and $1 - p = 0.058$.

(b) An urn contains eight blue balls and four green balls. Three balls are drawn from this urn without replacement. Compute the probability that all three balls are blue.

Solution: The probability of drawing a blue ball the first time is equal to 8/12. The probability of drawing a blue ball the second time given that the first ball is blue is 7/11. Finally, the probability of drawing a blue ball the third time given that the first two balls are blue is 6/10. Hence, the probability of drawing three blue balls equals the product $8/12 \cdot 7/11 \cdot 6/10 = 14/55$.

9. **[15 points]** Suppose X and Y are zero-mean unit-variance jointly Gaussian random variables.

(a) If $\rho_{X,Y} = 0.2$, find the numerical value of $E[Y|X = 5]$.
Solution: Since X and Y are jointly Gaussian,

$$E[Y|X = 5] = \hat{E}[Y|X = 5] = \rho_{X,Y} X = 1.$$

(b) If $\rho_{X,Y} = 0$, find $f_{Y|X}(v|u = 0)$ for all v.
Solution: Since X and Y are jointly Gaussian, $\rho_{X,Y} = 0$ implies that they are independent. Hence,

$$f_{Y|X}(v|u = 0) = f_Y(v) = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}.$$

10. **[15 points]** You are given two hypothesis, H_0 and H_1, and the corresponding conditional distributions of the observed random variable X given the hypotheses, as shown in the table. Here, $c \in (-\frac{1}{2}, \frac{1}{2})$ is some constant.

Table 1: The conditional distributions of X given H_0 and H_1, respectively.

X	0	1	2
H_0	$\frac{1}{2} + c$	$\frac{1}{2} - c$	0
H_1	0	$\frac{1}{2} + c$	$\frac{1}{2} - c$

(a) Describe the ML decision rule. Your answer will depend on the value of c.

Solution: Clearly, in the ML rule, when $X = 0$ is observed we should decide in favor of H_0 and when $X = 2$ we should decide in favor of H_1. The only question remains which decision to make when $X = 1$. Clearly,

$$\frac{P\{X = 1|H_1\}}{P\{X = 1|H_0\}} \ge 1$$

holds if and only if $c \in [0, \frac{1}{2})$, in which case we decide in favor of hypothesis H_1. Otherwise, when $c \in (-\frac{1}{2}, 0)$ we decide in favor of hypothesis H_0. Note that we could have broken the tie arbitrarily for $c = 0$ - in this case, we chose to break the tie in favor of hypothesis H_1.

(b) Let $c = 0$. Find the probability of miss, false alarm and average error probability of the ML estimator (when computing the average error probability, assume that the hypothesis are equally likely.)

Solution: Note that if $c = 0$ we may decide in favor of H_1 or H_0 while breaking the ties. We opt for the former. Clearly, we only make an error if $X = 1$ is observed, in which case we have $P_{fa} = P\{\text{Decide } H_1 | H_0\} = \frac{1}{2}$. Similarly, we have $P_{miss} = P\{\text{Decide } H_0 | H_1\} = 0$, which gives $P_{error} = \frac{1}{2}\frac{1}{2} = \frac{1}{4}$.

11. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Let T denote an exponentially distributed random variable with parameter λ.

TRUE	FALSE	
☐	☐	$P\{T \geq 0\} = e^{-\lambda}$.
☐	☐	$P\{T = 0\} = \lambda e^{-\lambda t}$.

Solution: False,False

(b) Consider a binary hypothesis testing problem with some known prior distribution (π_0, π_1). Let $p_{e,MAP}$ and $p_{e,ML}$ be the average probability of error of the MAP and ML rules, respectively.

TRUE	FALSE	
☐	☐	$p_{e,MAP} \leq p_{e,ML}$.
☐	☐	$p_{false-alarm} + p_{miss} \leq 1$ for the ML rule.

Solution: True, True

(c) Suppose X and Y are some jointly continuous random variables with finite variance.

TRUE	FALSE	
☐	☐	If X and Y are jointly Gaussian and uncorrelated, they must be independent.
☐	☐	If $Y = 2X + 5$, the MMSE of the unconstrained estimator of X given Y is equal to 0.
☐	☐	If $Y = X^2$, X and Y are correlated.

Solution: True,True,False

(d) Consider any three events, A, B and C, on a common probability space.

TRUE	FALSE	
☐	☐	$P(A \bigcup B \| C) \geq P(A \bigcup B)$.
☐	☐	If $P(AB\|C) = P(AB)$, then A, B and C are independent.
☐	☐	If A, B and C are independent, then $P(A\|C) = P(A\|B)$.

Solution: False,False,True