ECE 313 (Section G)
Homework 10
Due Date: Wed, Apr 26, in the class

Covariance and Correlation

Problem 1 – Suppose $n$ fair dice are independently rolled. Let:

$$X_k = \begin{cases} 1 & \text{if 1 shows on the } k^{th} \text{ die} \\ 0 & \text{else} \end{cases}$$

$$Y_k = \begin{cases} 1 & \text{if 2 shows on the } k^{th} \text{ die} \\ 0 & \text{else} \end{cases}$$

Let $X = \sum_{k=1}^{n} X_k$, which is the number of one’s showing, and $Y = \sum_{k=1}^{n} Y_k$, which is the number of two’s showing. Note that if a histogram is made recording the number of occurrences of each of the six numbers, then $X$ and $Y$ are the heights of the first two entries in the histogram.

a) What distribution best describes the random variables $X_k$ and $Y_k$?

b) Find $E[X_k]$ and $E[Y_k]$.

c) Use $E[X_k]$ and $E[Y_k]$ to find $E[X]$ and $Var(X)$.

d) Find $Cov(X_i, Y_j)$ if $1 \leq i \leq n$ and $1 \leq j \leq n$.

(Hint: Are $X_i$ and $Y_j$ independent when $i \neq j$? What if $i = j$?)

e) Use $Cov(X_i, Y_j)$ to find $Cov(X, Y)$.

f) Find the correlation coefficient $\rho_{X,Y}$. Are $X$ and $Y$ positively correlated, uncorrelated, or negatively correlated?

Problem 2 – Consider two random variables $X$ and $Y$:

a) If $Var(X + 2Y) = Var(X - 2Y)$, are $X$ and $Y$ uncorrelated? Are they independent? Why?

b) If $Var(X) = Var(Y)$, are $X$ and $Y$ uncorrelated? Why?

Hypothesis Testing

Problem 3 – Given the following likelihood matrix, specify the ML and MAP decision rules and find $p_{false-alarms}$, $p_{miss}$, and $p_{e}$ for both of them. Assume that: $\pi_0 = 0.3$.

<table>
<thead>
<tr>
<th></th>
<th>$X = 0$</th>
<th>$X = 1$</th>
<th>$X = 2$</th>
<th>$X = 3$</th>
<th>$X = 4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_1$</td>
<td>0.00</td>
<td>0.10</td>
<td>0.24</td>
<td>0.34</td>
<td>0.32</td>
</tr>
<tr>
<td>$H_0$</td>
<td>0.08</td>
<td>0.15</td>
<td>0.31</td>
<td>0.31</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Problem 4 – We are interested in detecting Denial of Service (DOS) attacks to a Web server based on the incoming traffic that is observed at a given second.
Suppose that usually the number of requests, \( X \), arriving at the Web Server from one client in an interval of 1 second is Poisson distributed with parameter \( \lambda = 3 \).

In case of a DOS attack, an attacker sends a total of 100 malicious requests per second, from which \( X \) would successfully pass the firewall and arrive to the Web Server. Suppose the probability that a malicious request passes through the firewall and arrives to the server is \( p = 0.04 \).

A decision rule is needed to decide, based on observation of \( X \), whether an attack occurred (hypothesis \( H_1 \)) or not (hypothesis \( H_0 \)).

a) Write the conditional pmf of observation \( X \) given hypotheses \( H_0 \).

b) Write the conditional pmf of observation \( X \) given hypotheses \( H_1 \). (\textbf{Hint:} What distribution best describes the number of malicious requests that successfully arrive to the Web Server?)

c) Describe the ML decision rule in the form of likelihood ratio tests (LRT).

(\textbf{Hint:} For which values of \( X \) an attack (\( H_1 \)) is declared by the ML decision rule?)

d) Describe the MAP decision rule in the form of likelihood ratio tests (LRT), under the assumption that receiving normal requests is a priori three times more likely than receiving attacks (i.e. \( \frac{\pi_0}{\pi_1} = 2 \)).

e) Find \( p_{\text{false alarm}}, p_{\text{miss}}, \) and \( p_e \) for each of the ML and MAP decision rules.

f) Compare the performance of the ML or MAP decision rules.