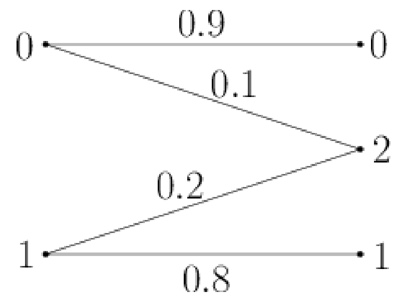
University of Illinois Spring 2017

ECE 313: Hour Exam II

Wednesday, April 12, 2017
8:00 p.m. — 9:15 p.m.
Last names A-T in ECEB 1002; U-Z in ECEB 1015.

1. **[5+3+2 points]**

The figure below shows a binary communication channel, called *binary erasure channel*, with binary input $X \sim \text{Bern}(1/3)$ (that is, $p_X(0) = 2/3$ and $p_X(1) = 1/3$) and ternary output $Y \sim \{0, 1, 2\}$.



The conditional pmf $p_{Y|X}(y|x)$ of Y given X is given in the figure; e.g., $p_{Y|X}(2|0) = 0.1$, $p_{Y|X}(0|1) = 0$, $p_{Y|X}(2|1) = 0.2$, etc.

(a) Find $p_{X,Y}(x, y)$, $p_Y(y)$, and $p_{X|Y}(x|y)$ for all possible values of X and Y.

Solution: The joint pmf can be computed using the chain rule $p_{X,Y}(x,y) = p_{Y|X}(y|x)p_X(x)$:

$p_{X,Y}(0,0) = \frac{9}{10} \times \frac{2}{3} = \frac{18}{30}$
$p_{X,Y}(0,1) = 0 \times \frac{2}{3} = 0$
$p_{X,Y}(0,2) = \frac{1}{10} \times \frac{2}{3} = \frac{2}{30}$
$p_{X,Y}(1,0) = 0 \times \frac{1}{3} = 0$
$p_{X,Y}(1,1) = \frac{8}{10} \times \frac{1}{3} = \frac{8}{30}$
$p_{X,Y}(1,2) = \frac{2}{10} \times \frac{1}{3} = \frac{2}{30}$

The marginal pmf $p_Y(y)$ is found by the law of total probability $p_Y(y) = \sum_x p_{X,Y}(x,y)$:

$p_Y(0) = p_{X,Y}(0,0) + p_{X,Y}(1,0) = \frac{18}{30}$
$p_Y(1) = p_{X,Y}(0,1) + p_{X,Y}(1,1) = \frac{8}{30}$
$p_Y(2) = p_{X,Y}(0,2) + p_{X,Y}(1,2) = \frac{4}{30}$

Finally we use the definition of conditional probability $p_{X|Y}(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$

$p_{X|Y}(0|0) = 1$
$p_{X|Y}(0|1) = 0$
$p_{X|Y}(0|2) = \frac{1}{2}$
$p_{X|Y}(1|0) = 0$
$p_{X|Y}(1|1) = 1$
$p_{X|Y}(1|2) = \frac{1}{2}$

(b) A decoder receives the output symbol Y, and decides whether the input symbol was $X = 0$ or $X = 1$ by applying the following decision rule $D(y)$.

$$D(y) = \begin{cases} 0 & \text{if } p_{X|Y}(0|y) \geq p_{X|Y}(1|y) \\ 1 & \text{otherwise} \end{cases} \tag{1}$$

Find $D(0), D(1)$, and $D(2)$.
Solution:
$D(0) = 0$, since $p_{X|Y}(0|0) = 1 \geq p_{X|Y}(1|0) = 0$.
$D(1) = 1$, since $p_{X|Y}(0|1) = 0 < p_{X|Y}(1|1) = 1$.
$D(2) = 0$, since $p_{X|Y}(0|2) = 1/2 \geq p_{X|Y}(1|2) = 1/2$ (note that since $p_{X|Y}(0|2) = p_{X|Y}(1|2)$, we could have also chosen $D(2) = 1$).

(c) What is the probability that the decoder makes an error, that is, $P_e = P\{D(Y) \neq X\}$?
Solution:
The probability of error is

$$P_e = P\{D(Y) \neq X\} = \sum_y \sum_{x: x \neq D(y)} p_{X,Y}(x, y) = p_{X,Y}(1, 0) + p_{X,Y}(0, 1) + p_{X,Y}(1, 2) = \frac{2}{30}$$

2. **[4+3 points]** Suppose there are two hypotheses about an observation X as follows:
H_0 : X has uniform distribution over $[0, c]$,
H_1 : X has exponential distribution with parameter λ,
where the constants c and λ are not known to us. An experiment is run under each hypotheses to first estimate these constants. X is observed under hypothesis zero for three times independently and these are the outputs: $0.2, 1.1$, and 2. Separately, X is observed under hypothesis one for three times independently and these are the outputs: $0.9, 1.6$, and 2.5.

(a) Based on the observations we have, what is the ML estimation of constants c and λ?
Solution: The likelihood function of observing $0.2, 1.1$, and 2 for hypothesis 0 is the following:

$$l_0(c) = \frac{1}{c} I_{\{0 \leq 0.2 \leq c\}} \cdot \frac{1}{c} I_{\{0 \leq 1.1 \leq c\}} \cdot \frac{1}{c} I_{\{0 \leq 2 \leq c\}} = \frac{1}{c^3} I_{\{0 \leq (0.2 \ \& \ 1.1 \ \& \ 2) \leq c\}},$$

where $\hat{c} = 2$ maximizes this likelihood function.
The likelihood function of observing $0.9, 1.6, 2.5$ for hypothesis 1 is the following:

$$l_1(\lambda) = \lambda e^{-0.9\lambda} \cdot \lambda e^{-1.6\lambda} \cdot \lambda e^{-2.5\lambda} = \lambda^3 e^{-5\lambda}.$$

The following is the derivative of $l_1(\lambda)$ with respect to λ:

$$\frac{dl_1(\lambda)}{d\lambda} = 3\lambda^2 e^{-5\lambda} - 5\lambda^3 e^{-5\lambda} = \lambda^2 e^{-5\lambda}(3 - 5\lambda) = 0.$$

We should check which value of λ among both the end points for λ and the points derived from setting the derivative to zero, maximizes $l_1(\lambda)$. Checking the three points $0, \frac{3}{5}$, and ∞, we find out that $\hat{\lambda} = \frac{3}{5}$.

(b) Assuming $c = e$ and $\lambda = 1$, find the ML rule.
Solution: For $u \in [0, e]$:

$$\Lambda(u) = \frac{f_1(u)}{f_0(u)} = e \cdot e^{-u} = e^{-u+1},$$

then,

$$\Lambda \leq 1 \equiv e^{-u+1} \leq 1 \equiv 1 \leq u \leq e,$$

so

$$\text{ML rule} : \begin{cases} H_0, & \text{if } 1 \leq X \leq e \\ H_1, & \text{else.} \end{cases}$$

3. **[3+4+3 points]**
Consider random variables X and Y with joint pdf

$$f_{X,Y}(x,y) = \begin{cases} c & \text{if } |x|+|y| \le 1/\sqrt{2} \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

where $c > 0$ is a constant.

(a) Find c.
Solution:
The density is constant over the diamond with corners $(0, \pm\sqrt{2}/2)$ and $(\pm\sqrt{2}/2, 0)$. Since the sides of the diamond have length equal to 1, its area is equal to 1, and $c = 1$. The value of c can also be obtained computing the double integral of the joint pdf.

(b) Find $f_X(x)$ and $f_{X|Y}(x|y)$. Are X and Y independent?
Solution: The marginal pdf of X can be found by integrating the joint pdf with respect to y.
For $0 \le x \le \sqrt{2}/2$,

$$f_X(x) = \int_{-\frac{1}{\sqrt{2}}+x}^{\frac{1}{\sqrt{2}}-x} 1dy = 2(\frac{1}{\sqrt{2}} - x)$$

For $-\sqrt{2}/2 \le x \le 0$,

$$f_X(x) = \int_{-\frac{1}{\sqrt{2}}-x}^{\frac{1}{\sqrt{2}}+x} 1dy = 2(\frac{1}{\sqrt{2}} + x)$$

Combining both results, we can express $f_X(x)$ as:

$$f_X(x) = \begin{cases} \sqrt{2} - 2|x| & \text{if } |x| \le \sqrt{2}/2 \\ 0 & \text{otherwise} \end{cases}$$

Note that $f_Y(y) = f_X(y)$ since $f_{X,Y}(x,y)$ is symmetric in x and y. Therefore,

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \begin{cases} \frac{1}{\sqrt{2}-2|y|} & \text{if } |x|+|y| \le \sqrt{2}/2, \\ 0 & \text{otherwise} \end{cases}$$

Clearly, X and Y are not independent, as the support for the density is not a product set $((0,0), (7/10, 0), (0, 7/10)$ are within the support, but $(7/10, 7/10)$ is without.

(c) Find $E[|X| + |Y|]$.
Solution:
First note that by symmetry, $E[|X| + |Y|] = 2E[|X|]$.

$$E[|X| + |Y|] = 2E[|X|] = 2 \times 4 \int_0^{1/\sqrt{2}} \int_0^{1/\sqrt{2}-x} xdydx = \frac{2}{3\sqrt{2}}$$

4. **[2+4+3 points]** Let $(N_t : t \ge 0)$ be a Poisson process with parameter $\lambda = 1$.

(a) Find the value of $P\Big(N_2 = 1 \Big| N_1 = 2\Big)$.
Solution: Zero!

(b) Find the value of $P\Big(N_3 \le 2 \Big| N_1 \le 1\Big)$.

Solution:

$$P\Big(N_3 \le 2 \Big| N_1 \le 1\Big) = \frac{P\big(N_3 \le 2 \;\&\; N_1 \le 1\big)}{P\big(N_1 \le 1\big)}.$$

The denominator is calculated as follows:

$$P\big(N_1 \le 1\big) = e^{-1} + e^{-1} = 2e^{-1},$$

and the numerator is calculated as below:

$$\begin{aligned} &P\big(N_3 \le 2 \;\&\; N_1 \le 1\big) \\ &= P\big(N_1 = 0 \;\&\; N_3 - N_1 \le 2\big) + P\big(N_1 = 1 \;\&\; N_3 - N_1 \le 1\big) \\ &= e^{-1} \cdot (5e^{-2}) + e^{-1} \cdot (3e^{-2}) = 8e^{-3}. \end{aligned}$$

Hence,

$$P\Big(N_3 \le 2 \Big| N_1 \le 1\Big) = 4e^{-2}$$

(c) As you know, (N_t) has independent increments, i.e., if $0 \le t_0 \le t_1 \le \cdots \le t_n$, the increments $N_{t_1} - N_{t_0},\ N_{t_2} - N_{t_1}, \cdots, N_{t_n} - N_{t_{n-1}}$ are independent.

Define $(Y_t : t \ge 0)$ where $Y_t = N_t^2$. Express $\mathbb{E}\left[Y_2 - Y_1 \Big| N_1\right]$ in terms of N_1.

Deduce from the expression you obtained for $\mathbb{E}\left[Y_2 - Y_1 \Big| N_1\right]$ whether $(Y_2 - Y_1)$ is independent of $(Y_1 - Y_0)$.

Solution: Conditioned on $N_1 = k$, and using the fact that $N_2 - N_1$ is an expeonentially $(\lambda = 1)$ distributed random variable independent of N_1, one has

$$\mathbb{E}(N_2^2 - N_1^2 | N_1 = k) = \sum_{l=0}^{\infty} \mathbb{E}(N_2^2 - N_1^2 | N_1 = k, N_2 - N_1 = l)\mathbb{P}(N_2 - N_1 = l) = \sum_{l=0}^{\infty} ((l+k)^2 - k^2)e^{-1}/l!$$

Simplifying this sum, one gets

$$e^{-1} \sum_{l=1}^{\infty} (l^2 - 2kl)/l! = e^{-1} \left(\sum_{l=1}^{\infty} l/(l-1)! + \sum_{l=1}^{\infty} 2k/(l-1)! \right).$$

Renaming $l - 1 =: m$, we obtain

$$e^{-1} \left(\sum_{m=0}^{\infty} (m+1)/m! + 2k \sum_{m=0}^{\infty} 1/m! \right) = e^{-1} \sum_{m=1}^{\infty} 1/(m-1)! + e^{-1} \sum_{m=0}^{\infty} 1/m! + e^{-1} 2k \sum_{m=0}^{\infty} 1/m! = 2 + 2k.$$

Therefore, $\mathbb{E}(N_2^2 - N_1^2 | N_1 = k) = 2 + 2k$, or, $\mathbb{E}(N_2^2 - N_1^2 | N_1) = 2 + 2N_1$.
As you see, the increment depends on N_1, or equivalently $(Y_1 - Y_0)$. Therefore, $(Y_2 - Y_1)$ is not independent of $(Y_1 - Y_0)$ and (Y_t) does not have independent increments.

5. **[3+4 points]** Let $X \sim N(0, 1)$ be the standard normal variable. Let $Y = \exp(X)$.

(a) Find $\mathbb{P}(Y > 0), \mathbb{P}(Y > 1)$.
Solution: Clearly, $Y > 0$ with probability 1. Further, $\mathbb{P}(Y > 1) = \mathbb{P}(X > 0) = 1/2$.

(b) Find the expectation and the variance of Y.

Solution: We find

$$\mathbb{E}Y^k = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ku} e^{-u^2/2} du = \frac{1}{\sqrt{2\pi}} e^{k^2/2} \int_{-\infty}^{\infty} e^{-k^2/2+ku-u^2/2} du = e^{k^2/2}$$

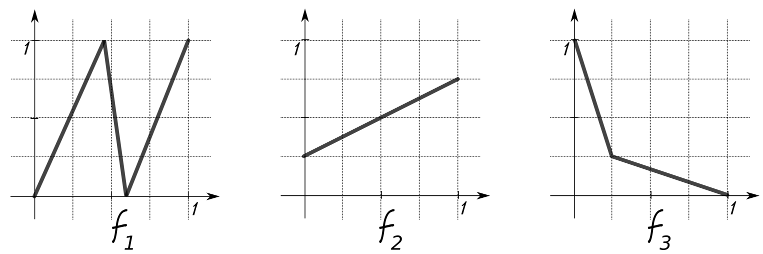
(here we use the fact that $\int_{-\infty}^{\infty} e^{-k^2/2+ku-u^2/2} du = \int_{-\infty}^{\infty} e^{-(u-k)^2/2} du$ and change of variable $u \mapsto u - k$).

Therefore,

$$\mathbb{E}(Y) = e^{1/2}, \text{and } \mathbb{E}Y^2 = e^2,$$

whence $\sigma_Y^2 = e^2 - e$.

6. **[5+4 points]** Consider the following 3 functions, described by their piece-wise linear graphs:



(a) If U is uniformly distributed on $[0, 1]$, describe (or sketch) the densities of $f_i(U), i = 1, 2, 3$.

Solution:

- For each $c \in [0, 1]$, there are 3 preimages of c. At these preimages, the absolute values of the reciprocals to the derivative of f_1 are always the same and sum up to 1. Hence $f_1(U)$ is also uniform in $[0, 1]$.
  Alternatively, one can evaluate CDF of $f_1(U)$. One can easily see that the total length of the set $\{u \in [0, 1] : f_1(u) \leq c\}$ is increasing linearly with $0 \leq c \leq 1$, starting at 0 for $c = 0$, and ending at 1 when $c = 1$. Hence $f_1(U)$ is uniform on $[0, 1]$.
- The density of $f_2(U)$ is uniform in its support, which is clearly equal $[1/4, 3/4]$ (same reasonings as before work).
- The density will be equal to 3 on $[0, 1/4]$ (reciprocal to the derivative to f_3 on the preimage), and to $1/3$ on $[1/4, 1]$, for the same reason.

(b) Find the expected value and variance of $f_i(f_i(U))$ for $i = 1, 2$.

Solution:

- For f_1, all iterations are uniform on $[0, 1]$, so the average is $1/2$, and the variance is $1/12$.
- For f_2, the iterations are uniform on the interval shrinking geometrically to $1/2$; the second iteration is uniform on $[1/2 - 1/8, 1/2 + 1/8]$. In other words, the $f_i(f_i(U))$ is distributed as $1/2 + 1/4(U - 1/2)$. The expectation is, therefore, $1/2$, and the variance $1/12 \times (1/4)^2 = 1/192$.