University of Illinois Spring 2017

ECE 313: Final Exam

Monday, May 8, 2017
7:00 p.m. — 10:00 p.m.
Last names A-T in ECEB 1002; U-Z in ECEB 1015.

1. **[2+3+4 points]**
   Let X be a real random variable, and F_X its CDF. Assume that

$$F_X(k) = 1 - 2^{-k}, k = 0, 1, 2, \ldots.$$

   (a) What are the probabilities that $X < 0$? that $X > 4$?
   Solution: From definition, those are 0, $2^{-4} = 1/16$.

   (b) i. Can the probability $\mathbb{P}(X = 3)$ be strictly positive?
   ii. Be greater than $1/10$?
   iii. Be greater than $1/8$?
   iv. What is the largest possible value for the probability $1 \leq X \leq 3$?
   Solution:
   i. Yes. For example, if X is geometrically distributed with $p = 1/2$, $\mathbb{P}(X = 3) = 1/8$.
   ii. Ditto.
   iii. No: $\mathbb{P}(X = 3) = F_X(3) - F_X(3-)$, and $F_X(3-) \geq F_X(2) = 3/4$. So $\mathbb{P}(X = 3) \leq 7/8 - 3/4 = 1/8$.
   iv. Similarly, $\mathbb{P}(1 \leq X \leq 3) \leq 7/8$, which is attained for the same geometric r.v.

   (c) If Y is an exponential r.v. with parameter 2, i.e. $\mathbb{P}(Y > c) = e^{-2c}, c \geq 0$, and $Z = e^{-Y}$, what is the CDF of Z?
   Solution: Clearly, $0 \leq Z \leq 1$ with probability 1. Further, $\mathbb{P}(Z < c) = \mathbb{P}(e^{-Y} < c) = \mathbb{P}(Y > -\log c) = e^{2\log c} = c^2$. Hence, $F_Z(c) = c^2$ on $[0, 1]$.

2. **[3+3+4 points]**
   The "birthday paradox" concerns the probability that, in a set of n randomly chosen people, at least two people will have birthdays the same day. Assuming a year always has 365 days and that each of those days is equally likely for a birthday, it can be shown that this probability is more than 50% with just 23 people, and more than 99.9% with 70 people. This seems counter-intuitive, but it is completely true, hence the name paradox. In this problem we will compute this probability exactly.

   Let's denote by A the event that at least two people in a set of n have the same birthday. We assume that these n people are chosen at random, and that a year has always 365 days, each of them equally likely for a birthday.

   (a) Find $P(A)$, when $n > 365$ and $n = 1$.
   Solution: If $n > 365$, there must be two people who have birthdays the same day, since a year only has 365 days, and thus $P(A) = 1$ in this case. If $n = 1$, on the other hand, $P(A) = 0$, since there is only one person.

   (b) Find $P(A)$, when $n = 2$.
   Solution: If $n = 2$, both people need to have birthdays the same day. Since there are 365 possibilities for the birthday, and each person will have that birthday with probability $1/365$, we can conclude that

$$P(A) = \binom{365}{1}\left(\frac{1}{365}\right)^2 = \frac{1}{365}.$$

(c) Find $P(A)$, for $1 < n < 366$.

Solution: It is easier to compute $P(A^c)$, that is, the probability that none of the n people have birthdays the same day. This implies that all n people should have birthdays at different days. And since the the probability of having a birthday on a given day is 1/365, we have that

$$P(A^c) = n!\binom{365}{n}\left(\frac{1}{365}\right)^n = \frac{365!}{(365-n)!}\left(\frac{1}{365}\right)^n$$

And therefore we conclude that

$$P(A) = 1 - P(A^c) = 1 - \frac{365!}{(365-n)!}\left(\frac{1}{365}\right)^n$$

3. **[3+4 points]** The two parts of this problem are unrelated.

(a) Let $P(A) = 0.7, P(B^c) = 0.4$ and $P(A \cup B) = 0.7$. Find $P(A^c|B^c)$ and $P(B^c|A)$.

Solution:

$$P(A^c|B^c) = \frac{P(A^cB^c)}{P(B^c)} = \frac{P((A\cup B)^c)}{P(B^c)} = \frac{1-0.7}{0.4} = \frac{3}{4}$$

$$P(B^c|A) = 1-P(B|A) = 1-\frac{P(AB)}{P(A)} = 1-\frac{P(A)+P(B)-P(A\cup B)}{P(A)} = 1-\frac{0.7+0.6-0.7}{0.7} = \frac{1}{7}$$

(b) Let X be a Geometric random variable with parameter p. Find $P_X(k|X \in A) = P\{X = k|X \in A)$, that is, the conditional pmf of X given that X belongs to A, where A is the following event: $A = \{X \text{ is an even number}\}$.

Solution:

$$P\{X = k|X \in A\} = \frac{P\{X = k, X \in A\}}{P\{X \in A\}}$$

Note that for odd k, $P\{X = k, X \in A\} = 0$, and thus $P\{X = k|X \in A\} = 0$.
For even k, we have:

$$\begin{aligned}
P\{X = k|X \in A\} &= \frac{P\{X = k, X \in A\}}{P\{X \in A\}} \\
&= \frac{P\{X \in A|X = k\}P\{X = k\}}{P\{X \in A\}} \\
&= \frac{P\{X = k\}}{P\{X \in A\}} \\
&= \frac{(1-p)^{k-1}p}{\sum_{k=1}^{\infty}(1-p)^{2k-1}p} \\
&= \frac{(1-p)^{k-1}p}{\sum_{k=0}^{\infty}(1-p)^{2k+1}p} \\
&= \frac{(1-p)^{k-1}p}{p(1-p)\sum_{k=0}^{\infty}((1-p)^2)^k} \\
&= \frac{(1-p)^{k-1}p(1-(1-p)^2)}{p(1-p)} \\
&= (1-p)^{k-2}(1-(1-p)^2)
\end{aligned}$$

4. **[5+5 points]** You have two bags of M&Ms, let's call them Bag A and Bag B. Bag A contains the following proportion of colors: 20% blue, 10% green, 15% brown, 30% yellow, and 25% red, whereas Bag B contains the following proportion of colors: 20% green, 20% brown, 25% yellow, 30% red, and 5% tan.

   (a) You pick one of the bags at random, and then you start taking M&Ms out from the bag randomly, until you get 5 green M&Ms. Let X denote the number of M&Ms you pick randomly until you get 5 green ones. We have the following two hypothesis related to observation X.
   H_0: The M&Ms were picked from Bag A.
   H_1: The M&Ms were picked from Bag B.
   Find the ML rule for the possible values of X (no need to determine exact values of X where your rule flips, just write down explicit conditions on it).
   Solution: The probability that X takes value k is

$$P\{X = k\} = \binom{k-1}{4} p^5 (1-p)^{k-5},$$

   where p is the probability of selecting a green M&M. Note that this equation is valid only for $k \geq 5$, since for $k < 5$ the probability is zero.
   Under hypothesis H_0, the probability of selecting a green M&M is 0.1, whereas under hypothesis H_1 it is 0.2.
   Therefore, the ML rule is

$$\begin{aligned}\Lambda(k) &= \frac{\binom{k-1}{4} 0.2^5 (1-0.2)^{k-5}}{\binom{k-1}{4} 0.1^5 (1-0.1)^{k-5}} \\ &= \frac{0.2^5 (0.8)^{k-5}}{0.1^5 (0.9)^{k-5}} \\ &= \left(\frac{18}{8}\right)^5 \left(\frac{8}{9}\right)^k\end{aligned}$$

   Thus we will select H_1 for those values of k such that $\Lambda(k) = (\frac{18}{8})^5 (\frac{8}{9})^k \geq 1$.

   (b) You now select one of the bags at random, and pick from it one M&M at random, which happens to be green. Then you take another M&M at random from the other bag, which happens to be yellow. What is the probability that the green M&M came from Bag A? (Hint: Use Bayes formula)
   Solution: Let's define the following events:
   A: The first bag you selected is Bag A, and the other one is Bag B.
   B: The first bag you selected is Bag B, and the other one is Bag A.
   E: The green M&M comes from the first bag, and the yellow one from the other one.
   Thus we are interested in computing the probability $P(A|E)$.
   Using Bayes formula, we get

$$\begin{aligned}P(A|E) &= \frac{P(AE)}{P(E)} \\ &= \frac{P(E|A)P(A)}{P(E|A)P(A) + P(E|B)P(B)},\end{aligned}$$

   with $P(A) = P(B) = 1/2$, $P(E|A) = 0.1 \times 0.25$, and $P(E|B) = 0.3 \times 0.2$.
   Thus

$$\begin{aligned}P(A|E) &= \frac{0.1 \times 0.25 \times 0.5}{0.1 \times 0.25 \times 0.5 + 0.3 \times 0.2 \times 0.5} \\ &= \frac{5}{17}\end{aligned}$$

5. **[3+3+4 points]** Consider two servers giving service to two tasks, in parallel. The service times of the first and second tasks are exponentially distributed with parameters λ_1 and λ_2, denoted by X_1 and X_2, respectively. The service times of the two tasks are independent from each other.

   (a) What is $P(X_1 > 5|X_1 > 3)$?
   Solution: Using the memory-less property of geometric random variables,

$$P(X_1 > 5|X_1 > 3) = P(X_1 > 2) = e^{-2\lambda_1}$$

   (b) What is $\mathbb{E}[X_1|X_1 > 3]$?
   Solution: Using the memory-less property of geometric random variables,

$$\mathbb{E}[X_1|X_1 > 3] = 3 + \mathbb{E}[X] = 3 + \frac{1}{\lambda_1}$$

   (c) What is $P(X_1 > X_2)$?
   Solution:

$$P(X_1 > X_2) = \int_0^\infty \lambda_1 e^{-\lambda_1 u} \int_0^u \lambda_2 e^{-\lambda_2 v} \, dvdu = \frac{\lambda_2}{\lambda_1 + \lambda_2}.$$

6. **[4+3 points]**

   (a) Consider the random variables X, Y with densities

$$f_X(u) = \begin{cases} 1/2 & \text{for } u \in [0,1] \cup [2,3] \\ 0 & \text{elsewhere} \end{cases}$$

$$f_Y(u) = \begin{cases} 1/4 & \text{for } u \in [0,4] \\ 0 & \text{elsewhere} \end{cases}$$

   Find the support and the maximum of the density function of $X + Y$.
   Solution: The support is evident: $[0, 7]$. The maxima can be obtained by inspection, maximizing the overlap of the densities and is equal to $1/4$.

   (b) If the joint distribution of the random values (X, Y) is uniform in the disk $\{u^2+v^2 \leq 1\}$, find the values of the cumulative distribution function $F(-1,-1), F(2,0), F(\sqrt{2}/2, \sqrt{2}/2)$.
   Solution: The density is $1/\pi$, and by computing the areas of the intersections of the disk with the third orthant displaced to the corresponding point, we find results $0, 1/2, 1/2 + 1/\pi$.

7. **[5+2+6 points]**

   Consider N random variables $X_k, k = 1, \ldots, N$. Assume that $\mu_{X_k} = 0, \sigma^2_{X_k} = 1, k = 1, \ldots, N$, and the correlation coefficients are all identical, $\rho_{X_k X_l} = \rho, k \neq l$.

   (a) Find the variance of $S = X_1 + X_2 + \ldots + X_N$.
   Solution: Using the linearity of the covariance, we obtain that

$$\sigma^2_S = \sum_{k=1}^N \sigma^2_{X_k} + \sum_{1 \leq k \neq l \leq N} \rho_{X_k X_l} = N + N(N-1)\rho.$$

(b) Find the correlation coefficient between $X_1 - X_2$ and $X_3 - X_4$.
Solution: By linearity, it is 0.

(c) Using Chebyshev inequality, estimate probability that

$$|A| \geq 1000,$$

for $A = X_1 - X_2 + X_3 - X_4 + \ldots + X_{N-1} - X_N$, where $N = 20,000$ and $\rho = .5$.
Solution: The key is to estimate the variance of the alternating sum A. As A is the sum of $N/2$ uncorrelated random variables $X_{2k-1} - X_{2k}$, its variance is $N/2$ times the variance of each of them, i.e.

$$\sigma_A^2 = N(1-\rho),$$

whence

$$\mathbb{P}(|A| > a) \leq \frac{\sigma_A^2}{a^2} = .01.$$

8. **[3+3+4 points]** Suppose X and Y have a joint distribution with $\mathbb{E}[X] = 1$, $\mathbb{E}[Y] = 2$ and $Var(X) = 1$. Also suppose that X and $X + Y$ are independent of each other.

(a) What is $Cov(X, Y)$?
Solution: X and $X + Y$ are independent, so

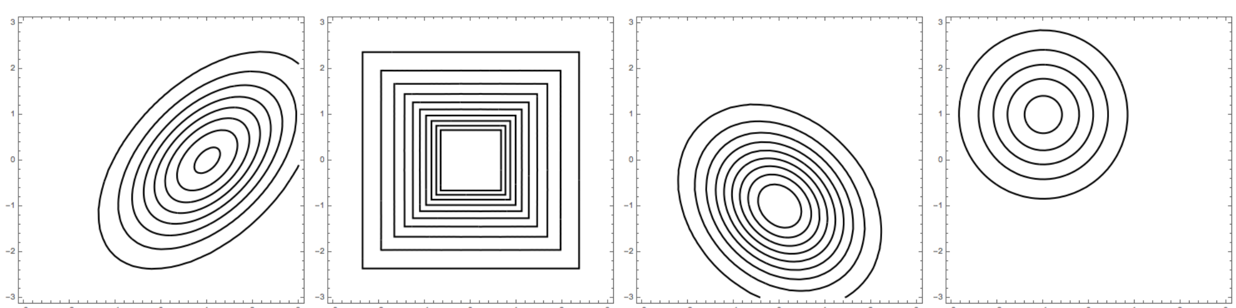
$$0 = Cov(X, X+Y) = Cov(X, X) + Cov(X, Y) = 1 + Cov(X, Y) \Rightarrow Cov(X, Y) = -1.$$

(b) What is $\mathbb{E}[X|(X+Y)^2 = 2]$?
Solution: X and $X + Y$ are independent, so X and $(X + Y)^2$ are also independent, so

$$\mathbb{E}[X|(X+Y)^2 = 2] = \mathbb{E}[X] = 1$$

(c) What is $\mathbb{E}[Y|X = 2]$?
Solution: We have $E[Y|X = 2] = E(X+Y-X|X = 2) = E(X+Y|X = 2) - E(X|X = 2)$. As $X + Y$ and X are independent, $E(X + Y|X = 2) = E(X + Y) = \mu_Y + \mu_X = 3$, so that $E[Y|X = 2] = 3 - E(X|X = 2) = 3 - 2 = 1$.

9. **[4+5+5 points]** Consider the contour plots of the densities:



(a) Match the densities below to their plots (write the letter under the corresponding plot):

$$\begin{array}{lll} A & f_1 = & K_1 \exp(-.5(|x+y| + |x-y|)) \\ B & f_2 = & K_2 \exp(-.5(x^2 - xy + y^2 - 2x + y)) \\ C & f_3 = & K_3 \exp(-.5(x^2 + y^2 + 2x - 2y)) \\ D & f_4 = & K_4 \exp(-.5(x^2 + .5xy + y^2 + .5x + 2y)) \end{array}$$

(here K_i are some normalization constants).

Solution: which matches second plot to A. Further, it is easy to find the peaks (modes) of the remaining densities by settind partial derivatives with respect to x, y to vanish. This would identify first plot with B, third plot with D and the last plot with C.

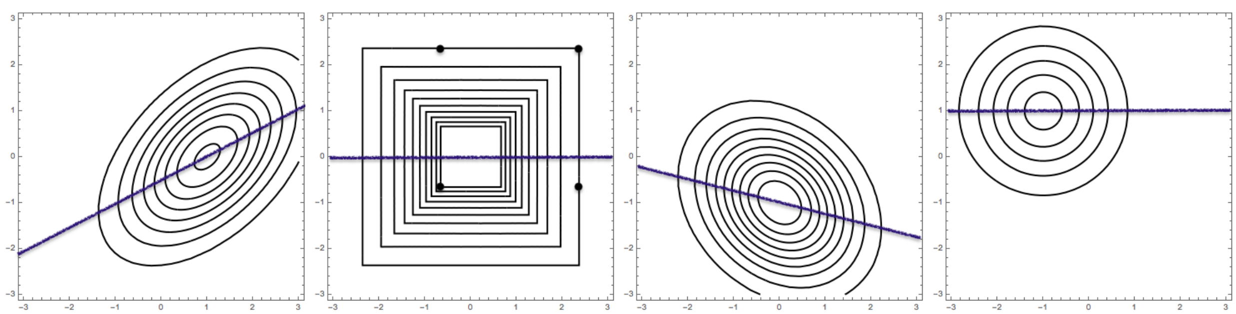
(b) Which of the distribution correspond to positively correlated, negatively correlated, uncorrelated pairs X, Y? For which of the distributions X, Y are independent?

Solution: Knowing formulae, it is easy to see that f_2 is positively, f_4 negatively correlated, and f_3 is uncorrelated. For f_1, we just note that the density is symmetric w.r.t. flip $u \leftrightarrow -u$, hence the expectations of XY and $(-X)Y$ are the same and therefore vanish. Further, both X, Y are centered. So, the covariance is 0, and the variables are uncorrelated as well.

Clearly, for f_4, as Gaussian distribution, uncorrelatedness implies independence, and in cases of f_2, f_3 the variables X, Y are dependent. For f_1, the densities at four fat points on the plot below witness the lack of independence.

(c) Sketch on the plots above the graphs of optimal linear MMSEs. Which of them are also the optimal unconstrained MMSEs? Write your answers below.

Solution:



The graphs should be linear functions going through the modes of conditional densities $f_{Y|X}$ for the Gaussian ones. This gives us cases B-D (here, the optimal linear is also the optimal unconstrained estimator).

For A, as the distribution survives flip $Y \leftrightarrow -Y$, the conditional density is symmetric for all $X = u$, and thus the optimal linear estimator is constant. Automatically, it is also the optimal unconstrained estimator.