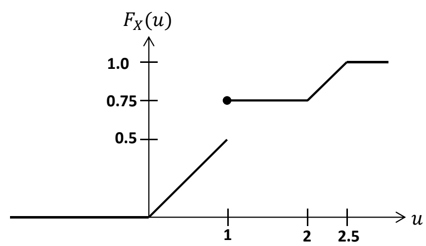
University of Illinois Spring 2016

ECE 313: Exam II

Wednesday, April 13, 2016
7:00 p.m. — 8:15 p.m.
Aa-Hs in DCL 1320,
Ht-Lf in ECEB 1013,
Lg-Np in ECEB 1015,
Nq-Zz in ECEB 1002

1. **[10 points]** Consider the CDF in the figure below.



Carefully provide numerical answers for the following; partial credit will not be assigned for incorrect answers.

(a) Obtain $P\{0.5 < X \le 1\}$.
Solution: Recalling that $F_X(c) = P\{X \le c\}$, one obtains $P\{0.5 < X \le 1\} = F_X(1) - F_X(0.5) = 0.75 - 0.25 = 0.50$.

(b) Obtain $P\{0.5 \le X < 1\}$.
Solution: $P\{0.5 \le X < 1\} = F_X(1^-) - F_X(0.5^-) = 0.5 - 0.25 = 0.25$.

2. **[20 points]** An amplifier has gain $A = gR$ where g, the transconductance, is a positive constant, and R, the load resistance, is uniformly distributed in the interval $[9, 11]$.

(a) Determine $E[A]$ and $\mathrm{Var}(A)$ in terms of g.
Solution: $E[A] = gE[R] = g\frac{(11+9)}{2} = 10g$, $\mathrm{Var}(A) = g^2\mathrm{Var}(R) = g^2\frac{(11-9)^2}{12} = \frac{g^2}{3}$.

(b) Determine the minimum value of g that would ensure that $P\{A \ge 10\} \ge 0.8$, i.e., at least 80% of the amplifiers have a gain of 10 or more.
Solution: Ensuring $P\{A \ge 10\} \ge 0.8$ is the same as ensuring $P\{gR \ge 10\} \ge 0.8$, or equivalently, $P\left\{R \ge \frac{10}{g}\right\} \ge 0.8$. We want the minimum g, so we solve $P\left\{R \ge \frac{10}{g}\right\} = 0.8$ for g. Since R is uniformly distributed over $[9, 11]$, this requires $\frac{10}{g} \in [9, 11]$ so that $P\left\{R \ge \frac{10}{g}\right\} = \left(11 - \frac{10}{g}\right)/2$. Solving $\left(11 - \frac{10}{g}\right)/2 = 0.8$ yields $g = 10/9.4 = 50/47$.

3. **[18 points]** The three parts of this problem are unrelated.

(a) Let X be Gaussian with mean 1 and variance 4. Express $P\{(X-1)^2 < 16\}$ in terms of the Φ function.
Solution: $P\{(X-1)^2 < 16\} = P\{-4 < X - 1 < 4\} = P\{\frac{-4}{2} < \frac{X-1}{2} < \frac{4}{2}\} = \Phi(2) - \Phi(-2)$, because $\frac{X-1}{2} \sim N(0, 1)$.

(b) Let $X \sim Binomial(160, 1/4)$. Use the Gaussian approximation with continuity correction to approximate $P\{45 \leq X \leq 50\}$ in terms of the Φ function.

Solution: Note that X has mean $np = 40$ and variance $np(1-p) = 30$, and under the Gaussian approximation with the continuity correction, $P\{45 \leq X \leq 50\}$
$= P\{44.5 \leq X \leq 50.5\} = P\{\frac{4.5}{\sqrt{30}} \leq \frac{X-40}{\sqrt{30}} \leq \frac{10.5}{\sqrt{30}}\} \approx \Phi\left(\frac{10.5}{\sqrt{30}}\right) - \Phi\left(\frac{4..5}{\sqrt{30}}\right)$.

(c) Let X have pdf $f_X(u) = \frac{u^2}{3}$ for $u \in (-2, 1)$ and zero else. Let $Y = -2X$ and obtain the pdf of Y, $f_Y(v)$ for all v.

Solution: Recall that if $Y = aX + b$, then $f_Y(v) = \frac{1}{|a|} f_X\left(\frac{v-b}{a}\right)$ (this also follows from the three step procedure for finding the pdf of a function of a random variable), hence in this case $f_Y(v) = \frac{1}{|-2|} f_X\left(\frac{v}{-2}\right) = \frac{1}{2} f_X\left(-\frac{v}{2}\right) = \frac{1}{6}\left(-\frac{v}{2}\right)^2 = \frac{v^2}{24}$. The support of f_Y is obtained by realizing the transformation of the endpoints of the support of f_X, which yields the left endpoint $u = -2$ transformed into $v = 4$ and the right endpoint $u = 1$ transformed into $v = -2$. Hence, $f_Y(v) = \frac{v^2}{24}$ for $v \in (-2, 4)$ and zero else.

4. **[18 points]** Suppose the times of tweets sent out by a certain celebrity form a Poisson process, with mean rate λ tweets per day.

(a) What is the probability no tweets are sent in a two day period? (Hint: Your answer should depend on λ.)

Solution: The number of tweets sent in two days has the Poisson distribution with mean 2λ, so the probability that number is zero is $e^{-2\lambda}$. (That is, $\frac{(2\lambda)^k e^{-2\lambda}}{k!}$ evaluated at $k = 0$.)

(b) How many days would it take, on average, for 20 tweets to be sent? (Hint: Your answer should depend on λ.)

Solution: The mean time until a tweet occurs is $1/\lambda$. So the time til 20 tweets occur has mean $20/\lambda$. This is the mean of the Erlang distribution with parameters λ and 20.

(c) Suppose the first day of observation there are 5 tweets, the second day 1 tweet, and the third day 2 tweets. What is $\widehat{\lambda}_{ML}$, the value of the maximum likelihood estimate of λ?

Solution: The three observations are independent, Poisson random variables, of mean λ. So the likelihood the observed values are 5,1,2 is $\frac{e^{-\lambda}\lambda^5}{5!}\frac{e^{-\lambda}\lambda}{1!}\frac{e^{-\lambda}\lambda^2}{2!}$, and the ML estimate is obtained by maximizing this likelihood with respect to λ. It is equivalent to maximizing $\lambda^8 e^{-3\lambda}$ with respect to λ. This has derivative $\lambda^7(8-3\lambda)e^{-3\lambda}$, which is zero for $\lambda = 8/3$. Moreover, the derivative is positive for $\lambda < 8/3$ and negative for $\lambda > 8/3$. So the maximizer is $8/3$. That is, $\widehat{\lambda}_{ML} = 8/3$.

5. **[18 points]** Suppose under hypothesis H_1, the observation X has the $N(3, 4)$ distribution. Under hypothesis H_0, the observation X has the $N(1, 4)$ distribution.

(a) Find the MAP decision rule in case the prior probabilities satisfy $\frac{\pi_0}{\pi_1} = e$ (where e is the base of natural logarithm, $e = 2.7182\cdots$.)

Solution: The likelihood ratio is given by:

$$\begin{aligned} \Lambda(u) &= \frac{f_1(u)}{f_0(u)} \\ &= \exp\left\{-\frac{(u-3)^2}{8} + \frac{(u-1)^2}{8}\right\} \\ &= \exp\left\{\frac{u-2}{2}\right\}. \end{aligned}$$

The MAP rule is to compare the likelihood ratio to the threshold $\frac{\pi_0}{\pi_1} = e$. We have

$$\Lambda(u) > e \text{ is equivalent to } \frac{u-2}{2} > 1 \text{ which is equivalent to } u > 4.$$

Hence the MAP rule is the following:
If $X > 4$, declare H_1 is true. If $X < 4$, declare H_0 is true.

(b) Find a decision rule such that $p_{false\ alarm} = p_{miss}$.
Solution: Since f_1 and f_0 are Gaussian pdfs with the same variance, they are each symmetric about their means, and they have the same shape. Thus, the decision rule that compares X to a threshold half way between the means gives $p_{false_alarm} = p_{miss}$. Since $\frac{3+1}{2} = 2$ the decision rule is the same as the ML rule: If $X > 2$, declare H_1 is true. If $X < 2$, declare H_0 is true.

6. **[16 points]** The following two parts are unrelated.

(a) Suppose X and Y have joint pdf $f_{X,Y}(u,v) = \begin{cases} Cuv & u^2+v^2 \leq 1, u \geq 0, v \geq 0 \\ 0 & \text{else,} \end{cases}$, for some appropriate constant C. Find the conditional pdf of Y given X, $f_{Y|X}(v|u)$, *and also determine whether X and Y are independent.* (Hints: It is not necessary to determine C. Be sure to specify what values of u $f_{Y|X}(v|u)$ is defined for, and specify for all $v \in \mathbb{R}$.)
Solution: The support of f_X is $(0,1)$. For $0 < u < 1$,
$f_X(u) = \int_0^{\sqrt{1-u^2}} Cuv dv = \frac{Cuv^2}{2}\Big|_{v=0}^{\sqrt{1-u^2}} = \frac{Cu(1-u^2)}{2}$.
Thus, $f_{Y|X}(v|u)$ is well defined if $0 < u < 1$, and for such values of u,

$$f_{Y|X}(v|u) = \begin{cases} \frac{Cuv}{\frac{Cu(1-u^2)}{2}} = \frac{2v}{1-u^2} & 0 \leq v \leq \sqrt{1-u^2} \\ 0 & \text{else.} \end{cases}$$

No, X and Y are not independent because the support of $f_{X,Y}$ is not a product set, or because $f_{Y|X}(v|u)$ depends on u.

(b) Suppose X and Y have joint pdf $f_{X,Y}(u,v) = \begin{cases} 0.5 & u \in [0,2], v \in [0,1] \\ 0 & \text{else.} \end{cases}$. Find the conditional pdf of Y given X, $f_{Y|X}(v|u)$, *and also determine whether X and Y are independent.*
Solution: Yes, X and Y are independent. X is uniformly distributed over $[0,2]$ and Y is uniformly distributed over $[0,1]$. Thus, $f_{Y|X}(v|u) = f_Y(v)$ for all v (if $u \in [0,2]$). That is, if $u \in [0,2]$,

$$f_{Y|X}(v|u) = \begin{cases} 1 & v \in [0,1] \\ 0 & v \notin [0,1]. \end{cases}$$