**University of Illinois** Spring 2015

# ECE 313: Final Exam

Monday, May 11, 2015
1:30 p.m. — 4:30 p.m.

**Name: (in BLOCK CAPITALS)** ________________________________

**NetID:** ________________________________

**Signature:** ________________________________

**Section:**
☐ E, 9:00 a.m. ☐ C, 10:00 a.m. ☐ D, 11:00 a.m. ☐ F, 1:00 p.m. ☐ B, 2:00 p.m.

Instructions

This exam is closed book and closed notes except that two 8.5”×11” sheets of notes is permitted: both sides may be used. No electronic equipment (cell phones, etc.) allowed.

The exam consists of 11 problems worth a total of 200 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading

1. 18 points ______
2. 20 points ______
3. 10 points ______
4. 15 points ______
5. 24 points ______
6. 21 points ______
7. 24 points ______
8. 20 points ______
9. 18 points ______
10. 30 points ______

Total (200 points) ______

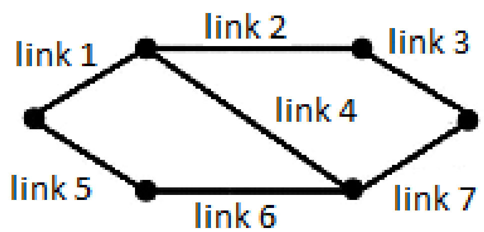
1. [**18 points**] You are in a gambling game, which has four overlapping events: A, B, C and D. For all parts of this problem, $P(A) = \frac{1}{2}$.

   (a) $P(AB) = \frac{1}{5}$. Events $A$ and $B$ are independent. What is $P(A^c B^c)$?

   (b) $P(C) = \frac{1}{2}$, and $P(AC) = \frac{1}{6}$. What is $P(A^c C^c)$?

   (c) $P(A|D) = \frac{2}{3}$. What is the maximum possible values of $P(D)$?

2. **[20 points]** Consider the following s-t network, where link $i \in \{2, \cdots, 6\}$ fails with probability $\frac{1}{2}$, independent of one another, and links 1 and 7 never fail. Let $c_i$ be the capacity of link $i$, with $c_1 = 30$, $c_2 = 10$, $c_3 = 15$, $c_4 = 20$, $c_5 = 20$, $c_6 = 15$, $c_7 = 15$.




(a) What values can the capacity of this network take?

(b) What is the outage probability?

3. [**10 points**] Consider the random variables $(X, Y)$ such that

$$(X,Y) = \begin{cases} (1,4) & \text{with probability } 1/3; \\ (2,2) & \text{with probability } 1/3; \\ (4,1) & \text{with probability } 1/3. \end{cases}$$

Find the value of CDF $F_{X,Y}$ at points

(a) $(2.5, 0.5)$

(b) $(2.5, 2.5)$

(c) $(1.5, 5.5)$

(d) $(3.5, 4.5)$

4. [**15 points**] The pdf for the r.v. $(X,Y)$ is given by

$$f_{X,Y}(x,y) = \begin{cases} 1/8 & \text{if} \quad \lfloor x \rfloor + \lfloor y \rfloor \text{ is even and } -2 \le x \le 2; -2 \le y \le 2, \\ 0 & \text{otherwise} \end{cases}$$

(here $\lfloor x \rfloor$ is the floor function, the largest integer $\le x$).

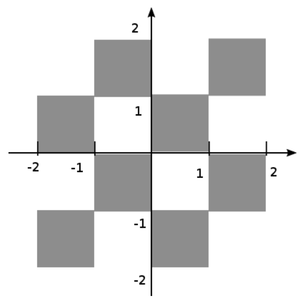



Figure 1: Probability density function $f_{X,Y}$.

(a) Find the marginal pdf for $X$.

(b) Find $\mathbb{P}(|X - Y| \le 1)$.

5. **[24 points]** Alice, Bob and Charlie are tossing fair coins at the same time, but independently of each other. Let $N$ be the number of these groups of simultaneous tosses until the first time *at least* one of them gets a HEAD on their toss.

   (a) Find the probability that $N$ is strictly larger than 2.

   (b) Calculate $E[N]$.

   (c) Find $E[N^2]$.

   (d) Calculate $E[N^2|N > 2]$.

6. [**21 points**] $X$ and $Y$ are two random variables with the following properties: $E[X] = 1$, $E[Y] = 3$, $\text{Var}(X) = 4, \text{Var}(Y) = 1, \rho_{X,Y} = 0.2$.

(a) If $Z = 2X + 3Y - 1$ then find $E[Z]$ and $\text{Var}(Z)$.

(b) Find the optimal *linear* estimate of $X$ given that $Y = 1$.

(c) Suppose $W = X + \alpha Y$. Find the value of $\alpha$ such that $W$ and $X$ are uncorrelated.

7. [**24 points**] Let $X$ and $Y$ be jointly Gaussian random variables. It is known that $\mu_Y = -\frac{2}{9}$ and $\sigma_Y^2 = 4$. It is also known that $\mu_{Y|X} = -\frac{1}{3}X$ and that the resulting minimum mean squared error for the best linear estimator is $MMSE_{\hat{E}[Y|X]} = 3$.

(a) Obtain the conditional pdf $f_{Y|X}(v|u)$ for all $v$ and $u$.

(b) Obtain the best unconstrained MSE estimator of $Y$ from $X$.

(c) Obtain $\hat{E}[Y|X = 3]$.

(d) Obtain $\mu_X$ and $\sigma_X^2$.

8. **[20 points]** Let $N_t$ be a Poisson process with rate 2.

(a) Let $0 < s < t$, obtain $E[N_t N_{t-s}]$.

(b) Obtain $P\{N_1 = 2 \text{ and } N_3 = 5\}$.

9. **[18 points]** Recall that if $X$ is a random variable with mean $\mu$ and variance $\sigma^2$, then $Y = \frac{X-\mu}{\sigma}$ defines the standardized version of $X$. For each of the following two choices of distribution for $X$, sketch and carefully label the pmfs of $X$ and $Y$.

   (a) $X$ is the number generated by rolling a fair die. Carefully sketch the pmf of $X$ and the pmf of $Y$. (Hint: The standard deviation of $X$ is given by $\sigma \approx 1.7$.)

   (b) $X$ has the binomial distribution with parameters $n = 4$ and $p = 0.5$. Carefully sketch the pmf of $X$ and the pmf of $Y$.

10. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) Suppose $X$ and $Y$ are independent, continuous-type random variables. $W = \max(X, Y)$, $Z = \min(X, Y)$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $F_W(t) = F_X(t)F_Y(t)$ |
| ☐ | ☐ | $F_Z(t) = (1 - F_X(t))(1 - F_Y(t))$ |
| ☐ | ☐ | $f_Z(t) = f_X(t)(1 - F_Y(t)) + f_Y(t)(1 - F_X(t))$ |

(b) Let $A$, $B$, $C$ be independent events with $0 < P(A), P(B), P(C) < 1$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $P(AC\|B) = P(AC\|B^c)$ |
| ☐ | ☐ | $P(AB\|C) \leq P(C\|AB)P(AB)$ |

(c) Let $X, Y$ be two independent normal random variables with $\mu_X = \mu_Y$ and with $\sigma_X > \sigma_Y$.

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $\mathbb{P}(Y \leq X) > 1/2$. |
| ☐ | ☐ | $\mathbb{P}(X = Y) > 0$ |

(d) $X$ and $Y$ are jointly distributed discrete random variables. They are uncorrelated if:

| TRUE | FALSE | |
|---|---|---|
| ☐ | ☐ | $\mathrm{Var}(X + Y) = \mathrm{Var}(X - Y)$ |
| ☐ | ☐ | E[XY] $= 0$ |
| ☐ | ☐ | $P(X = u, Y = v) = P(X = u)P(Y = v)$ for every pair $(u, v)$ |