University of Illinois Spring 2015

ECE 313: Hour Exam II

Wednesday, April 15, 2015
7:00 p.m. — 8:15 p.m.

1. **[14 points]** Let c be a constant and X be a random variable with pdf

$$f_X(u) = \begin{cases} \frac{1}{9}u^2 & u \in [0, c], \\ 0 & \text{else.} \end{cases} .$$

   (a) Obtain the value of c in order for $f_X(u)$ to be a valid pdf.
   Solution: The pdf has to be non-negative and it has to integrate to one. It is clearly non-negative and

$$1 = \int_{\infty}^{\infty} f_X(u)du = \int_0^c \frac{1}{9}u^2 du = \frac{c^3}{27},$$

   so that $c = (27)^{1/3} = 3$.

   (b) Determine the CDF $F_X(u)$ for all u.
   Solution: By definition, $F_X(u) = P\{X \leq u\}$. From the support of f_X we can clearly see that $F_X(u) = 0$ if $u < 0$ and $F_X(u) = 1$ if $u > 3$. If $u \in [0, 3]$ then

$$F_X(u) = \int_{-\infty}^{u} f_X(v)dv = \int_0^u \frac{1}{9}u^2 du = \frac{u^3}{27}.$$

   So that

$$F_X(u) = \begin{cases} 0 & u < 0, \\ \frac{u^3}{27} & u \in [0, 3], \\ 1 & u > 3 \end{cases} ,$$

2. **[18 points]** Planes arrive at O'hare as a Poisson process at the rate of 50 planes per hour. Let N be the number of planes that arrive in the time period between 9am and 10am.

   (a) Calculate $E[N]$.
   Solution: N is a Poisson random variable with parameter 50 (which is $50 \cdot 1$). So $E[N] = 50$.

   (b) Calculate $E[N]$ conditioned on the fact that 120 planes arrive in the time period between 9am and 11am.
   Solution: Conditioned on the fact that 120 planes arrive in the time period between 9am and 11am, N is Binomial(120,1/2). So the conditional mean is 60.

   (c) Calculate the probability that exactly 3 planes arrive in the minute between 9am and 9:01am, conditioned on the fact that 120 planes arrive in the time period between 9am and 11am. *Hint*: Use the Poisson approximation.
   Solution: Denoting M to be the (random) number of planes that arrive in the minute between 9 and 9:01am. Then, conditioned on the fact that 120 planes arrive in the time period between 9am and 11am, M is Binomial(120,1/120) which can be approximated by Poisson(1). So $P(M = 3) \approx \frac{e^{-1}}{6}$.

3. **[20 points]** Let X and Y be two random variables with joint pdf

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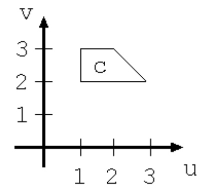
$$f_{X,Y}(u,v) = \begin{cases} c & 1 \le u \le 2,\ 2 \le v \le 3, \\ c & 2 \le u \le 3,\ 2 \le v \le 5-u, \\ 0 & \text{else.} \end{cases},$$



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where c is a constant.

(a) Determine the value of the constant c for $f_{X,Y}$ to be a valid joint pdf.
Solution: Clearly, X and Y are jointly uniform, so

$$f_{X,Y}(u,v) = \frac{1}{\text{area of support}} = \frac{1}{(1)^2 + \frac{(1)(1)}{2}} = \frac{2}{3} = c$$

because the support consists of a square of side length 1 and a triangle with base and height 1.

(b) Determine the marginal pdf of Y, $f_Y(v)$, for all v.
Solution: Clearly, from the support we can see that $f_Y(v) = 0$ if $v < 2$ or if $v > 3$. If $v \in [2,3]$ then we obtain the marginal by integrating over u:

$$f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)du = \int_1^{5-v} \frac{2}{3} du = \frac{2}{3}(4-v).$$

So,

$$f_Y(v) = \begin{cases} \frac{2}{3}(4-v) & 2 \le v \le 3, \\ 0 & \text{else.} \end{cases},$$

(c) Determine the conditional pdf $f_{X|Y}(u|v)$ for all u, v.
Solution: Obviously, if $f_Y(v) = 0$ then $f_{X|Y}(u|v)$ is undefined because we cannot divide by zero, and that occurs if $v \notin [2,3]$. If $v \in [2,3]$ then $f_{X|Y}(u|v) = \frac{f_{X,Y}(u,v)}{f_Y(v)} = \frac{\frac{2}{3}}{\frac{2}{3}(4-v)} = \frac{1}{4-v}$ if $u \in [1, 5-v]$ and zero else. Hence

$$f_{X|Y}(u|v) = \begin{cases} \text{undefined} & v \notin [2,3], \\ \frac{1}{4-v} & v \in [2,3], u \in [1, 5-v], \\ 0 & \text{else.} \end{cases},$$

4. **[16 points]** The two parts of this problem are unrelated.

(a) Let X be uniformly distributed in $[0,1]$. Find the CDF for

$$Y = 2|X - 1/2|.$$

Solution: Clearly, Y takes values between 0 and 1. For $0 \le y \le 1$,

$$\mathbf{P}(Y \le y) = \mathbf{P}(1/2 - y/2 \le X \le 1/2 + y/2) = y,$$

so that Y is uniform on $[0,1]$.

(b) Let Z be a random variable with pdf

$$f_Z(u) = \begin{cases} 0 & u < 2, \\ \frac{1}{2} & u \in [2,4], \\ 0 & u > 4. \end{cases},$$

Determine a function $q(Z)$ such that the random variable $W = q(Z)$ is uniformly distributed on $[0,1]$.

Solution: Notice that Z is uniformly distributed on $[2,4]$ so $q(Z)$ can be the linear transformation given by the CDF of Z, that is $q(Z) = F_Z(Z) = \frac{Z}{2} - 1$.

5. **[14 points]** The two parts of this problem are unrelated.

(a) Consider the following binary hypothesis testing problem. If hypothesis H_0 is true, the continuous random variable X is uniformly distributed on $(-1,1)$, while if hypothesis H_1 is true, the pdf of X is $f_1(u) = \begin{cases} 1-|u|, & |u| < 1, \\ 0, & \text{otherwise.} \end{cases}$

Find the decision region Γ_0 for the MAP (*maximum a posteriori probability*) decision rule if $\pi_0 = 1/4$. Remember that Γ_0 is the set of all real numbers such that if $X \in \Gamma_0$, the decision is that H_0 is the true hypothesis.

Solution: Recall that the likelihood ratio is given by $\Lambda(X = u) = \frac{f_1(u)}{f_0(u)}$. If $u \notin [-1,1]$ then $\Lambda(X = u)$ is undefined. Otherwise, $\Lambda(X = u) = 2(1 - |u|)$.

The MAP rule compares $\Lambda(X = u)$ to the ratio $\frac{\pi_0}{\pi_1} = \frac{1}{3}$ and decides for H_1 if $\Lambda(X = u) > \frac{1}{3}$, which ocurrs if $u \in \left(-\frac{5}{6}, \frac{5}{6}\right)$.

Hence, $\Gamma_0 = \left\{u \in \mathsf{R} : |u| \in \left[\frac{5}{6}, 1\right]\right\}$.

(b) Let X be a Gaussian random variable with mean μ and variance 4. Given that it is known that $P(X \geq 6) \approx 0.0228$, find the value of μ.

Solution: Standardizing X we have $X = \mu + 2N$ and $P(N \geq \frac{6-\mu}{2}) \approx 0.0228$. Using the Q tables we see that $P(N \geq 2) \approx 0.0228$, so we infer that $\mu = 2$.

6. **[18 points]** Let X and Y be two random variables with joint pdf

$$f_{X,Y}(u,v) = \begin{cases} e^{-(u+v)} & u \geq 0,\ v \geq 0, \\ 0 & \text{else.} \end{cases}$$

(a) Are X and Y independent? Indicate why or why not.

Solution: Yes, X and Y are independent since $e^{-(u+v)} = e^{-u}e^{-v}$ and the support is a product set. In fact, X and Y are two independent exponential random variables with mean 1.

(b) Let $Z = X + Y$. Find the CDF of Z.

Solution:

$$F_Z(a) = \int_0^a \int_0^{a-u} e^{-(u+v)} dv du = 1 - (1+a)e^{-a}.$$

(c) Find $P\{X > Y\}$.

Solution: The line $Y = X$ bisects the first quadrant, which is the support of $f_{X,Y}$, and the pdf is symmetric on u and v so $P\{X > Y\} = 1/2$. Alternatively,

$$P\{X > Y\} = \int_{-\infty}^{\infty} \int_{-\infty}^{u} f_{X,Y}(u,v) dv du = \int_0^{\infty} \int_0^{u} e^{-(u+v)} dv du = 1/2.$$

Table 6.2: Q function, the area under the standard normal pdf to the right of x.

x	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	0.5000	0.4960	0.4920	0.4880	0.4840	0.4801	0.4761	0.4721	0.4681	0.4641
0.1	0.4602	0.4562	0.4522	0.4483	0.4443	0.4404	0.4364	0.4325	0.4286	0.4247
0.2	0.4207	0.4168	0.4129	0.4090	0.4052	0.4013	0.3974	0.3936	0.3897	0.3859
0.3	0.3821	0.3783	0.3745	0.3707	0.3669	0.3632	0.3594	0.3557	0.3520	0.3483
0.4	0.3446	0.3409	0.3372	0.3336	0.3300	0.3264	0.3228	0.3192	0.3156	0.3121
0.5	0.3085	0.3050	0.3015	0.2981	0.2946	0.2912	0.2877	0.2843	0.2810	0.2776
0.6	0.2743	0.2709	0.2676	0.2643	0.2611	0.2578	0.2546	0.2514	0.2483	0.2451
0.7	0.2420	0.2389	0.2358	0.2327	0.2296	0.2266	0.2236	0.2206	0.2177	0.2148
0.8	0.2119	0.2090	0.2061	0.2033	0.2005	0.1977	0.1949	0.1922	0.1894	0.1867
0.9	0.1841	0.1814	0.1788	0.1762	0.1736	0.1711	0.1685	0.1660	0.1635	0.1611
1.0	0.1587	0.1562	0.1539	0.1515	0.1492	0.1469	0.1446	0.1423	0.1401	0.1379
1.1	0.1357	0.1335	0.1314	0.1292	0.1271	0.1251	0.1230	0.1210	0.1190	0.1170
1.2	0.1151	0.1131	0.1112	0.1093	0.1075	0.1056	0.1038	0.1020	0.1003	0.0985
1.3	0.0968	0.0951	0.0934	0.0918	0.0901	0.0885	0.0869	0.0853	0.0838	0.0823
1.4	0.0808	0.0793	0.0778	0.0764	0.0749	0.0735	0.0721	0.0708	0.0694	0.0681
1.5	0.0668	0.0655	0.0643	0.0630	0.0618	0.0606	0.0594	0.0582	0.0571	0.0559
1.6	0.0548	0.0537	0.0526	0.0516	0.0505	0.0495	0.0485	0.0475	0.0465	0.0455
1.7	0.0446	0.0436	0.0427	0.0418	0.0409	0.0401	0.0392	0.0384	0.0375	0.0367
1.8	0.0359	0.0351	0.0344	0.0336	0.0329	0.0322	0.0314	0.0307	0.0301	0.0294
1.9	0.0287	0.0281	0.0274	0.0268	0.0262	0.0256	0.0250	0.0244	0.0239	0.0233
2.0	0.0228	0.0222	0.0217	0.0212	0.0207	0.0202	0.0197	0.0192	0.0188	0.0183
2.1	0.0179	0.0174	0.0170	0.0166	0.0162	0.0158	0.0154	0.0150	0.0146	0.0143
2.2	0.0139	0.0136	0.0132	0.0129	0.0125	0.0122	0.0119	0.0116	0.0113	0.0110
2.3	0.0107	0.0104	0.0102	0.0099	0.0096	0.0094	0.0091	0.0089	0.0087	0.0084
2.4	0.0082	0.0080	0.0078	0.0075	0.0073	0.0071	0.0069	0.0068	0.0066	0.0064
2.5	0.0062	0.0060	0.0059	0.0057	0.0055	0.0054	0.0052	0.0051	0.0049	0.0048
2.6	0.0047	0.0045	0.0044	0.0043	0.0041	0.0040	0.0039	0.0038	0.0037	0.0036
2.7	0.0035	0.0034	0.0033	0.0032	0.0031	0.0030	0.0029	0.0028	0.0027	0.0026
2.8	0.0026	0.0025	0.0024	0.0023	0.0023	0.0022	0.0021	0.0021	0.0020	0.0019
2.9	0.0019	0.0018	0.0018	0.0017	0.0016	0.0016	0.0015	0.0015	0.0014	0.0014
3.0	0.0013	0.0013	0.0013	0.0012	0.0012	0.0011	0.0011	0.0011	0.0010	0.0010

x	0.0	0.2	0.4	0.6	0.8
0.0	0.5000000	0.4207403	0.3445783	0.2742531	0.2118553
1.0	0.1586553	0.1150697	0.0807567	0.0547993	0.0359303
2.0	0.0227501	0.0139034	0.0081975	0.0046612	0.0025552
3.0	0.0013500	0.0006872	0.0003370	0.0001591	0.0000724
4.0	0.0000317	0.0000134	0.0000054	0.0000021	0.0000008