1. **[14 points]** Let $c$ be a constant and $X$ be a random variable with pdf

$$f_X(u) = \begin{cases} \frac{1}{9}u^2 & u \in [0, c], \\ 0 & \text{else.} \end{cases}$$

(a) Obtain the value of $c$ in order for $f_X(u)$ to be a valid pdf.

**Solution:** The pdf has to be non-negative and it has to integrate to one. It is clearly non-negative and

$$1 = \int_{-\infty}^{\infty} f_X(u)du = \int_0^c \frac{1}{9}u^2du = \frac{c^3}{27},$$

so that $c = (27)^{1/3} = 3$.

(b) Determine the CDF $F_X(u)$ for all $u$.

**Solution:** By definition, $F_X(u) = P\{X \leq u\}$. From the support of $f_X$ we can clearly see that $F_X(u) = 0$ if $u < 0$ and $F_X(u) = 1$ if $u > 3$. If $u \in [0, 3]$ then

$$F_X(u) = \int_{-\infty}^{u} f_X(v)dv = \int_0^u \frac{1}{9}v^2dv = \frac{u^3}{27}.$$

So that

$$F_X(u) = \begin{cases} 0 & u < 0, \\ \frac{u^3}{27} & u \in [0, 3], \\ 1 & u > 3 \end{cases}.$$ 

2. **[18 points]** Planes arrive at O’hare as a Poisson process at the rate of 50 planes per hour. Let $N$ be the number of planes that arrive in the time period between 9am and 10am.

(a) Calculate $E[N]$.

**Solution:** $N$ is a Poisson random variable with parameter 50 (which is $50 \cdot 1$). So $E[N] = 50$.

(b) Calculate $E[N]$ conditioned on the fact that 120 planes arrive in the time period between 9am and 11am.

**Solution:** Conditioned on the fact that 120 planes arrive in the time period between 9am and 11am, $N$ is Binomial(120,1/2). So the conditional mean is 60.

(c) Calculate the probability that exactly 3 planes arrive in the minute between 9am and 9:01am, conditioned on the fact that 120 planes arrive in the time period between 9am and 11am. **Hint:** Use the Poisson approximation.

**Solution:** Denoting $M$ to be the (random) number of planes that arrive in the minute between 9 and 9:01am. Then, conditioned on the fact that 120 planes arrive in the time period between 9am and 11am, $M$ is Binomial(120,1/120) which can be approximated by Poisson(1). So $P(M = 3) \approx e^{-1}/6.$
3. [20 points] Let $X$ and $Y$ be two random variables with joint pdf

$$f_{X,Y}(u,v) = \begin{cases} 
c & 1 \leq u \leq 2, \, 2 \leq v \leq 3, \\
c & 2 \leq u \leq 3, \, 2 \leq v \leq 5 - u, \\
0 & \text{else.}
\end{cases}$$

where $c$ is a constant.

(a) Determine the value of the constant $c$ for $f_{X,Y}$ to be a valid joint pdf.

**Solution:** Clearly, $X$ and $Y$ are jointly uniform, so

$$f_{X,Y}(u,v) = \frac{1}{\text{area of support}} = \frac{1}{(1)^2 + (1)(1)} = \frac{2}{3} = c$$

because the support consists of a square of side length 1 and a triangle with base and height 1.

(b) Determine the marginal pdf of $Y$, $f_Y(v)$, for all $v$.

**Solution:** Clearly, from the support we can see that $f_Y(v) = 0$ if $v < 2$ or if $v > 3$. If $v \in [2,3]$ then we obtain the marginal by integrating over $u$:

$$f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)du = \int_1^{5-v} \frac{2}{3}du = \frac{2}{3}(4-v).$$

So,

$$f_Y(v) = \begin{cases} 
\frac{2}{3}(4-v) & 2 \leq v \leq 3, \\
0 & \text{else.}
\end{cases}$$

(c) Determine the conditional pdf $f_{X|Y}(u|v)$ for all $u,v$.

**Solution:** Obviously, if $f_Y(v) = 0$ then $f_{X|Y}(u|v)$ is undefined because we cannot divide by zero, and that occurs if $v \notin [2,3]$. If $v \in [2,3]$ then $f_{X|Y}(u|v) = \frac{f_{X,Y}(u,v)}{f_Y(v)} = \frac{\frac{2}{3}(4-v)}{\frac{2}{3}(4-v)} = \frac{1}{v-2}$ if $u \in [1,5-v]$ and zero else. Hence

$$f_{X|Y}(u|v) = \begin{cases} 
\text{undefined} & v \notin [2,3], \\
\frac{1}{v-2} & v \in [2,3], u \in [1,5-v], \\
0 & \text{else.}
\end{cases}$$

4. [16 points] The two parts of this problem are unrelated.

(a) Let $X$ be uniformly distributed in $[0,1]$. Find the CDF for

$$Y = 2|X - 1/2|.$$ 

**Solution:** Clearly, $Y$ takes values between 0 and 1. For $0 \leq y \leq 1$,

$$P(Y \leq y) = P(1/2 - y/2 \leq X \leq 1/2 + y/2) = y,$$

so that $Y$ is uniform on $[0,1]$. 

2
(b) Let $Z$ be a random variable with pdf

$$f_Z(u) = \begin{cases} 
0 & u < 2, \\
\frac{1}{2} & u \in [2, 4], \\
0 & u > 4.
\end{cases}$$

Determine a function $q(Z)$ such that the random variable $W = q(Z)$ is uniformly distributed on $[0, 1]$.

**Solution:** Notice that $Z$ is uniformly distributed on $[2, 4]$ so $q(Z)$ can be the linear transformation given by the CDF of $Z$, that is $q(Z) = F_Z(Z) = \frac{Z - 2}{4 - 2}$.

5. **[14 points]** The two parts of this problem are unrelated.

(a) Consider the following binary hypothesis testing problem. If hypothesis $H_0$ is true, the continuous random variable $X$ is uniformly distributed on $(-1, 1)$, while if hypothesis $H_1$ is true, the pdf of $X$ is $f_1(u) = \begin{cases} 
1 - |u|, & |u| < 1, \\
0, & \text{otherwise}.
\end{cases}$

Find the decision region $\Gamma_0$ for the MAP (maximum a posteriori probability) decision rule if $\pi_0 = 1/4$. Remember that $\Gamma_0$ is the set of all real numbers such that if $X \in \Gamma_0$, the decision is that $H_0$ is the true hypothesis.

**Solution:** Recall that the likelihood ratio is given by $\Lambda(X = u) = \frac{f_1(u)}{f_0(u)}$. If $u \not\in [-1, 1]$ then $\Lambda(X = u)$ is undefined. Otherwise, $\Lambda(X = u) = 2(1 - |u|)$. The MAP rule compares $\Lambda(X = u)$ to the ratio $\frac{\pi_0}{\pi_1} = \frac{1}{2}$ and decides for $H_1$ if $\Lambda(X = u) > \frac{1}{2}$, which occurs if $u \in \left(-\frac{5}{6}, \frac{5}{6}\right)$. Hence, $\Gamma_0 = \{u \in \mathbb{R} : |u| \in \left[\frac{3}{6}, 1\right]\}$.

(b) Let $X$ be a Gaussian random variable with mean $\mu$ and variance $4$. Given that it is known that $P(X \geq 6) \approx 0.0228$, find the value of $\mu$.

**Solution:** Standardizing $X$ we have $X = \mu + 2N$ and $P(N \geq \frac{5-\mu}{2}) \approx 0.0228$. Using the $Q$ tables we see that $P(N \geq 2) \approx 0.0228$, so we infer that $\mu = 2$.

6. **[18 points]** Let $X$ and $Y$ be two random variables with joint pdf

$$f_{X,Y}(u,v) = \begin{cases} 
e^{-u+v} & u \geq 0, v \geq 0, \\
0 & \text{else}.
\end{cases}$$

(a) Are $X$ and $Y$ independent? Indicate why or why not.

**Solution:** Yes, $X$ and $Y$ are independent since $e^{-u+v} = e^{-u}e^{-v}$ and the support is a product set. In fact, $X$ and $Y$ are two independent exponential random variables with mean $1$.

(b) Let $Z = X + Y$. Find the CDF of $Z$.

**Solution:**

$$F_Z(a) = \int_0^a \int_0^{a-u} e^{-(u+v)} dvdu = 1 - (1 + a)e^{-a}.$$ 

(c) Find $P\{X > Y\}$.

**Solution:** The line $Y = X$ bisects the first quadrant, which is the support of $f_{X,Y}$, and the pdf is symmetric on $u$ and $v$ so $P\{X > Y\} = 1/2$. Alternatively,

$$P\{X > Y\} = \int_{-\infty}^{\infty} \int_{-\infty}^{u} f_{X,Y}(u,v) dvdu = \int_0^{\infty} \int_0^{u} e^{-(u+v)} dvdu = 1/2.$$
Table 6.2: $Q$ function, the area under the standard normal pdf to the right of $x$.

<table>
<thead>
<tr>
<th>$x$</th>
<th>0.00</th>
<th>0.01</th>
<th>0.02</th>
<th>0.03</th>
<th>0.04</th>
<th>0.05</th>
<th>0.06</th>
<th>0.07</th>
<th>0.08</th>
<th>0.09</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>0.5000</td>
<td>0.4960</td>
<td>0.4920</td>
<td>0.4880</td>
<td>0.4840</td>
<td>0.4801</td>
<td>0.4761</td>
<td>0.4721</td>
<td>0.4681</td>
<td>0.4641</td>
</tr>
<tr>
<td>0.1</td>
<td>0.4602</td>
<td>0.4562</td>
<td>0.4522</td>
<td>0.4483</td>
<td>0.4443</td>
<td>0.4404</td>
<td>0.4364</td>
<td>0.4325</td>
<td>0.4286</td>
<td>0.4247</td>
</tr>
<tr>
<td>0.2</td>
<td>0.4207</td>
<td>0.4168</td>
<td>0.4129</td>
<td>0.4090</td>
<td>0.4052</td>
<td>0.4013</td>
<td>0.3974</td>
<td>0.3936</td>
<td>0.3897</td>
<td>0.3859</td>
</tr>
<tr>
<td>0.3</td>
<td>0.3821</td>
<td>0.3783</td>
<td>0.3745</td>
<td>0.3707</td>
<td>0.3669</td>
<td>0.3632</td>
<td>0.3594</td>
<td>0.3557</td>
<td>0.3520</td>
<td>0.3483</td>
</tr>
<tr>
<td>0.4</td>
<td>0.3446</td>
<td>0.3409</td>
<td>0.3372</td>
<td>0.3336</td>
<td>0.3300</td>
<td>0.3264</td>
<td>0.3228</td>
<td>0.3192</td>
<td>0.3156</td>
<td>0.3121</td>
</tr>
<tr>
<td>0.5</td>
<td>0.3085</td>
<td>0.3050</td>
<td>0.3015</td>
<td>0.2981</td>
<td>0.2946</td>
<td>0.2912</td>
<td>0.2877</td>
<td>0.2843</td>
<td>0.2810</td>
<td>0.2776</td>
</tr>
<tr>
<td>0.6</td>
<td>0.2743</td>
<td>0.2709</td>
<td>0.2676</td>
<td>0.2643</td>
<td>0.2611</td>
<td>0.2578</td>
<td>0.2546</td>
<td>0.2514</td>
<td>0.2483</td>
<td>0.2451</td>
</tr>
<tr>
<td>0.7</td>
<td>0.2420</td>
<td>0.2389</td>
<td>0.2358</td>
<td>0.2327</td>
<td>0.2296</td>
<td>0.2266</td>
<td>0.2236</td>
<td>0.2206</td>
<td>0.2177</td>
<td>0.2148</td>
</tr>
<tr>
<td>0.8</td>
<td>0.2119</td>
<td>0.2090</td>
<td>0.2061</td>
<td>0.2033</td>
<td>0.2005</td>
<td>0.1977</td>
<td>0.1949</td>
<td>0.1922</td>
<td>0.1894</td>
<td>0.1867</td>
</tr>
<tr>
<td>0.9</td>
<td>0.1841</td>
<td>0.1814</td>
<td>0.1788</td>
<td>0.1762</td>
<td>0.1736</td>
<td>0.1711</td>
<td>0.1685</td>
<td>0.1660</td>
<td>0.1635</td>
<td>0.1611</td>
</tr>
<tr>
<td>1.0</td>
<td>0.1587</td>
<td>0.1562</td>
<td>0.1539</td>
<td>0.1515</td>
<td>0.1492</td>
<td>0.1469</td>
<td>0.1446</td>
<td>0.1423</td>
<td>0.1401</td>
<td>0.1379</td>
</tr>
<tr>
<td>1.1</td>
<td>0.1357</td>
<td>0.1335</td>
<td>0.1314</td>
<td>0.1292</td>
<td>0.1271</td>
<td>0.1251</td>
<td>0.1230</td>
<td>0.1210</td>
<td>0.1190</td>
<td>0.1170</td>
</tr>
<tr>
<td>1.2</td>
<td>0.1151</td>
<td>0.1131</td>
<td>0.1112</td>
<td>0.1093</td>
<td>0.1075</td>
<td>0.1056</td>
<td>0.1038</td>
<td>0.1020</td>
<td>0.1003</td>
<td>0.0985</td>
</tr>
<tr>
<td>1.3</td>
<td>0.0968</td>
<td>0.0951</td>
<td>0.0934</td>
<td>0.0918</td>
<td>0.0901</td>
<td>0.0885</td>
<td>0.0869</td>
<td>0.0853</td>
<td>0.0838</td>
<td>0.0823</td>
</tr>
<tr>
<td>1.4</td>
<td>0.0808</td>
<td>0.0793</td>
<td>0.0778</td>
<td>0.0764</td>
<td>0.0749</td>
<td>0.0735</td>
<td>0.0721</td>
<td>0.0708</td>
<td>0.0694</td>
<td>0.0681</td>
</tr>
<tr>
<td>1.5</td>
<td>0.0668</td>
<td>0.0655</td>
<td>0.0643</td>
<td>0.0630</td>
<td>0.0618</td>
<td>0.0606</td>
<td>0.0594</td>
<td>0.0582</td>
<td>0.0571</td>
<td>0.0559</td>
</tr>
<tr>
<td>1.6</td>
<td>0.0548</td>
<td>0.0537</td>
<td>0.0526</td>
<td>0.0516</td>
<td>0.0505</td>
<td>0.0495</td>
<td>0.0485</td>
<td>0.0475</td>
<td>0.0465</td>
<td>0.0455</td>
</tr>
<tr>
<td>1.7</td>
<td>0.0446</td>
<td>0.0436</td>
<td>0.0427</td>
<td>0.0418</td>
<td>0.0409</td>
<td>0.0401</td>
<td>0.0392</td>
<td>0.0384</td>
<td>0.0375</td>
<td>0.0367</td>
</tr>
<tr>
<td>1.8</td>
<td>0.0359</td>
<td>0.0351</td>
<td>0.0344</td>
<td>0.0336</td>
<td>0.0329</td>
<td>0.0322</td>
<td>0.0314</td>
<td>0.0307</td>
<td>0.0301</td>
<td>0.0294</td>
</tr>
<tr>
<td>1.9</td>
<td>0.0287</td>
<td>0.0281</td>
<td>0.0274</td>
<td>0.0268</td>
<td>0.0262</td>
<td>0.0256</td>
<td>0.0250</td>
<td>0.0244</td>
<td>0.0239</td>
<td>0.0233</td>
</tr>
<tr>
<td>2.0</td>
<td>0.0228</td>
<td>0.0222</td>
<td>0.0217</td>
<td>0.0212</td>
<td>0.0207</td>
<td>0.0202</td>
<td>0.0197</td>
<td>0.0192</td>
<td>0.0188</td>
<td>0.0183</td>
</tr>
<tr>
<td>2.1</td>
<td>0.0179</td>
<td>0.0174</td>
<td>0.0170</td>
<td>0.0166</td>
<td>0.0162</td>
<td>0.0158</td>
<td>0.0154</td>
<td>0.0150</td>
<td>0.0146</td>
<td>0.0143</td>
</tr>
<tr>
<td>2.2</td>
<td>0.0139</td>
<td>0.0136</td>
<td>0.0132</td>
<td>0.0129</td>
<td>0.0125</td>
<td>0.0122</td>
<td>0.0119</td>
<td>0.0116</td>
<td>0.0113</td>
<td>0.0110</td>
</tr>
<tr>
<td>2.3</td>
<td>0.0107</td>
<td>0.0104</td>
<td>0.0102</td>
<td>0.0099</td>
<td>0.0096</td>
<td>0.0094</td>
<td>0.0091</td>
<td>0.0089</td>
<td>0.0087</td>
<td>0.0084</td>
</tr>
<tr>
<td>2.4</td>
<td>0.0082</td>
<td>0.0080</td>
<td>0.0078</td>
<td>0.0075</td>
<td>0.0073</td>
<td>0.0071</td>
<td>0.0069</td>
<td>0.0068</td>
<td>0.0066</td>
<td>0.0064</td>
</tr>
<tr>
<td>2.5</td>
<td>0.0062</td>
<td>0.0060</td>
<td>0.0059</td>
<td>0.0057</td>
<td>0.0055</td>
<td>0.0054</td>
<td>0.0052</td>
<td>0.0051</td>
<td>0.0049</td>
<td>0.0048</td>
</tr>
<tr>
<td>2.6</td>
<td>0.0047</td>
<td>0.0045</td>
<td>0.0044</td>
<td>0.0043</td>
<td>0.0041</td>
<td>0.0040</td>
<td>0.0039</td>
<td>0.0038</td>
<td>0.0037</td>
<td>0.0036</td>
</tr>
<tr>
<td>2.7</td>
<td>0.0035</td>
<td>0.0034</td>
<td>0.0033</td>
<td>0.0032</td>
<td>0.0031</td>
<td>0.0030</td>
<td>0.0029</td>
<td>0.0028</td>
<td>0.0027</td>
<td>0.0026</td>
</tr>
<tr>
<td>2.8</td>
<td>0.0026</td>
<td>0.0025</td>
<td>0.0024</td>
<td>0.0023</td>
<td>0.0022</td>
<td>0.0021</td>
<td>0.0020</td>
<td>0.0019</td>
<td>0.0018</td>
<td>0.0017</td>
</tr>
<tr>
<td>2.9</td>
<td>0.0019</td>
<td>0.0018</td>
<td>0.0017</td>
<td>0.0016</td>
<td>0.0016</td>
<td>0.0015</td>
<td>0.0015</td>
<td>0.0014</td>
<td>0.0014</td>
<td>0.0014</td>
</tr>
<tr>
<td>3.0</td>
<td>0.0013</td>
<td>0.0013</td>
<td>0.0012</td>
<td>0.0012</td>
<td>0.0011</td>
<td>0.0011</td>
<td>0.0011</td>
<td>0.0010</td>
<td>0.0010</td>
<td>0.0010</td>
</tr>
</tbody>
</table>