University of Illinois Spring 2014

ECE 313: Conflict Final Exam

Tuesday, May 13, 2014, 7:00 p.m. — 10:00 p.m.
Room 241 Everitt Lab

Name: (in BLOCK CAPITALS) ________________________________

NetID: ______________________ **Signature:** ______________________

Section: ☐ E, 9:00 a.m. ☐ C, 10:00 a.m. ☐ D, 11:00 p.m. ☐ F, 1:00 p.m.

Instructions

This exam is closed book and closed notes except that two 8.5"×11" sheets of notes are permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

The exam consists of nine problems worth a total of 200 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

Grading	
1. 18 points	______
2. 24 points	______
3. 24 points	______
4. 24 points	______
5. 18 points	______
6. 14 points	______
7. 24 points	______
8. 24 points	______
9. 30 points	______
Total (200 points)	______

1. **[18 points]** Consider an experiment in which a fair coin is repeatedly tossed every ten seconds, with the first toss happening ten seconds after time zero. Assume the tosses are independent.

   (a) (4 points) Associated to the k^{th} toss, define a random variable X_k that takes the value 1 if the outcome is heads and 0 otherwise. What is the pmf of X_k?

   (b) (2 points) What is the name of the random process defined by the infinite sequence of random variables $X_1, X_2, \ldots$?

   (c) (6 points) Find the pmf for the total number of heads that appear in the first 10 minutes.

   (d) (6 points) Find the pmf for the time it takes until a heads shows for the first time.

2. **[24 points]** Let T_1 and T_2 denote two independent exponential random variables with the same parameter, λ.

   (a) (6 points) Find the pdf of $T_t = T_1 + T_2$.

   (b) (6 points) Find the pdf of $T_s = \min\{T_1, T_2\}$.

   (c) (6 points) Find the pdf of $T_p = \max\{T_1, T_2\}$.

   (d) (6 points) Without doing any calculation, determine which is larger, $E[T_s]$ or $E[T_p]$. Explain your reasoning.

3. **[24 points]** At a tennis tournament, Serena is up against an opponent who she never met before. The match is best of five sets, where the one who first wins three sets is the winner. There are three equally likely scenarios for Serena:

- She is a stronger player and she wins each set with probability $3/4$.
- She is a weaker player and she wins each set with probability $1/4$.
- She is equally good as her opponent and she wins each set with probability $1/2$.

Given which scenario holds, the outcomes of the sets are mutually independent.

(a) (6 points) What is the probability the match ends in three straight sets?

(b) (6 points) What is the probability Serena loses the match?

(c) (6 points) It turns out that Serena lost the match by 1-3. What was the probability for that to happen?

(d) (6 points) After the match Serena decides to reevaluate herself. Given that she lost by 1-3, what is the conditional probability Serena is a stronger player than her opponent?

4. **[24 points]** Suppose (X, Y) is uniformly distributed within the triangular region $\{(u, v) : 0 \leq v \leq u \leq 1\}$, i.e.,

$$f_{XY}(u, v) = \begin{cases} 2 & 0 \leq v \leq u \leq 1 \\ 0 & \text{else} \end{cases}$$

(a) (6 points) Are X and Y independent? Explain your answer.

(b) (6 points) Suppose we want to estimate Y from X in the way that minimizes the mean-square error (MSE). Find the best unconstrained estimator $g^*(X)$ and the corresponding MSE.

(c) (6 points) Find the best linear estimator $L^*(X)$ and the corresponding MSE.

(d) (6 points) Find the correlation coefficient $\rho_{X,Y}$.

5. **[18 points]** Consider a vehicle traveling along a straight road in the direction of increasing mile markers. Suppose that at time zero, the location of the vehicle U is uniformly distributed between zero and one (i.e. between mile marker zero and mile marker one), and suppose the speed of the vehicle V is a constant over time, which is exponentially distributed with parameter $\lambda > 0$, and is independent of U. Thus, the location of the vehicle as a function of time t (measured in hours) is given by $X_t = U + Vt$.

   (a) (6 points) Identify the joint pdf of (U, V). Be sure to specify the support (i.e. where the joint pdf is not zero.)

   (b) (6 points) Find the mean and variance of X_t for a fixed $t > 0$. Make your answer as explicit as possible.

   (c) (6 points) Let T be the time the vehicle reaches the one mile marker. Find the CDF of T.

6. **[14 points]** Suppose 192 tickets are sold for an airplane flight with 150 seats. Suppose each passenger arrives for the flight with probability $p = 0.75$.

   (a) (7 points) Assuming each passenger arrives independently with probability p, use the central limit theorem (CLT) to express the approximate probability the flight is oversold (i.e. strictly more than 150 passengers arrive for the flight) in terms of the Q function. You do *not* need to use the continuity correction.

   (b) (7 points) Suppose instead that the 192 passengers consist of 96 pairs of passengers, such that for each pair, both passengers in the pair arrive for the flight with probability $p = 0.75$; otherwise neither passenger in the pair arrives for the flight. Assume pairs arrive independently. The number of seats on the plane is still 150. Find the Gaussian approximation for the probability the flight is oversold.

7. **[24 points]** Suppose under hypothesis H_0, the observation X has density f_0, and under hypothesis H_1, the observation X has density f_1, where the densities are given by

$$f_0(u) = \begin{cases} \frac{1}{2} & |u| \leq 1 \\ 0 & |u| \geq 1 \end{cases} \qquad\qquad f_1(u) = \begin{cases} 1 - |u| & |u| \leq 1 \\ 0 & |u| \geq 1 \end{cases}$$

(a) (6 points) Describe the ML decision rule for deciding which hypothesis is true for observation X.

(b) (6 points) Find $p_{\text{false alarm}}$ and p_{miss} for the ML rule.

(c) (6 points) Suppose it is assumed a priori that H_0 is true with probability π_0 and H_1 is true with probability $\pi_1 = 1 - \pi_0$. For what values of π_0 does the MAP decision rule declare H_0 with probability one, no matter which hypothesis is really true?

(d) (6 points) For the prior distribution $\pi_0 = 0.2$ and $\pi_1 = 0.8$, find $P\left(H_0 \,\middle|\, |X| < 0.5\right)$.

8. **[24 points]** Suppose X has the pdf:

$$f_X(u) = \begin{cases} 0.5u & 0 \le u \le 2 \\ 0 & \text{else} \end{cases}$$

(a) (6 points) Find $E[X^2]$.

(b) (6 points) Find $P(\lfloor X^2 \rfloor) = 1)$, where $\lfloor u \rfloor$ is the greatest integer less than or equal to u.

(c) (6 points) Find the cumulative distribution function (CDF) of $Y = \ln X$.

(d) (6 points) If U is uniformly distributed over the interval $[0, 1]$, find a function g such that $g(U)$ has the same distribution as X.

9. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) This part concerns different choices for the joint pdf of random variables X and Y. The constant c stands for a normalizing constant in each part.

TRUE	FALSE	
☐	☐	X and Y are independent if $f_{X,Y}(u,v) = \begin{cases} ce^{uv} & 0 \le u \le 1, 0 \le v \le 1 \\ 0 & \text{else} \end{cases}$
☐	☐	X and Y are independent if $f_{X,Y}(u,v) = \begin{cases} ce^{uv} & u^2 + v^2 \le 1 \\ 0 & \text{else} \end{cases}$

(b) Suppose X is a *continuous* type random variable with CDF $F_X(u)$.

TRUE	FALSE	
☐	☐	If $b > a$, then $F_X(b) > F_X(a)$.
☐	☐	If $F_X(b) > F_X(a)$, then $b > a$.
☐	☐	$F_X(\alpha) = \frac{1}{4}$ for some α with $-\infty < \alpha < \infty$.

(c) Let (X,Y) be uniformly distributed over $[0,1] \times [0,1]$.

TRUE	FALSE	
☐	☐	The support of $Z = X - Y$ is $[-1, 1]$.
☐	☐	$Z = X + Y$ is uniformly distributed over $[0,2]$.

(d) Let $X_1, X_2, \ldots$ be an infinite sequence of independent Bernoulli random variables all with parameter $p \in (0,1)$.

TRUE	FALSE	
☐	☐	$C_{10} = \sum_{k=1}^{10} X_k$ has a geometric distribution with parameter p.
☐	☐	Let $C_m = \sum_{k=1}^{m} X_k$, then the mean of $C_m - C_{m-1}$ is p.
☐	☐	$X = C_{10} - C_8$, and $Y = C_9 - C_7$ are independent random variables.