University of Illinois Spring 2014

ECE 313: Hour Exam I

Monday, March 3, 2014
7:00 p.m. — 8:15 p.m.
228 Nat. History (Sec. E/9am and C/10am) & 314 Altgeld (D/11am and F/1pm)

1. (a) The solution is n^2. For each choice that Tom can make (for a total of n), Jerry also has n choices because Jerry can choose the same jar that Tom chooses.

   (b) The solution is $n(n-1)$. Tom has n choices, and for each choice Tom can make, Jerry has $n-1$ choices.

   (c) If $n \geq 10$ the condition is satisfied if the second jar selected is one of the four jars on either side of the first jar chosen. So for any choice of the first jar, the condition is satisfied for 8 out of $n-1$ choices of the second jar. The probability is thus $\frac{8}{n-1}$. The condition is satisfied with probability one if $2 \leq n \leq 9$.

2. (a) $X \sim \mathsf{Binomial}(n, \frac{1}{n})$; $p_X(k) = \binom{n}{k}\left(\frac{1}{n}\right)^k\left(\frac{n-1}{n}\right)^{n-k} = \binom{n}{k}\frac{(n-1)^{n-k}}{n^n}, 0 \leq k \leq n.$

   (b) $\mathsf{E}[2X+1] = 2\mathsf{E}[X] + 1 = 2n\frac{1}{n} + 1 = 3.$
   $\mathsf{var}[2X+1] = 4\mathsf{var}[X] = 4n\frac{1}{n}\left(1 - \frac{1}{n}\right) = \frac{4(n-1)}{n}.$

   (c) Poisson with unit mean. That is, $p_X(k) \approx \frac{e^{-1}}{k!}$ for $k \geq 0$.

   (d) By symmetry, it is $\frac{1}{2}$. To arrive at this answer without using symmetry, we could argue as follows. To be definite, suppose the balls can be distinguished. Let A denote the event that exactly one ball falls into the first bin and B denote the event that exactly one ball falls into the first two bins. Then $|A| = n(n-2)^{n-1}$, because there are n choices of which ball to put into the first bin and for each of those choices, $(n-2)^{n-1}$ choices of where to put the other $n-1$ balls. And $|B| = 2n(n-2)^{n-1}$, because there are n choices of which ball to put into either the first or second bin, two ways to choose which of those two bins to use, and then $(n-2)^{n-1}$ choices of where to put the other $n-1$ balls. So $P(A|B) = \frac{P(AB)}{P(B)} = \frac{|AB|}{|B|} = \frac{1}{2}$.

   (e) $P(A) = P(B) = (1 - \frac{1}{n})^n$ and, since each ball independently misses both the first and last bins with probability $1 - \frac{2}{n}$, $P(AB) = \left(1 - \frac{2}{n}\right)^n$. So $P(AB) \neq P(A)P(B)$. Therefore A and B are not independent.

   (f) There are $n!$ ways to place the balls into the bins so that every bin is not empty, and a total of n^n ways to place all the balls into the bins. So the probability is $\frac{n!}{n^n}$.

3. (a) The width of the original window is $\frac{a}{\sqrt{n}}$, where a determines the confidence level (which we don't need to find). To reduce this width by a factor of two, n should be increased by a factor of four. So 1200 samples would be needed.

   (b) We need to find ρ to maximize $(1-\rho)\rho^{10}$, or equivalently, $\rho^{10} - \rho^{11}$, which has derivative $10\rho^9 - 11\rho^{10} = (10 - 11\rho)\rho^9$. The derivative is zero at $\rho = 10/11$, and it is positive to the left of that value and negative to the right of that value. So $\widehat{\rho}_{ML}(10) = \frac{10}{11}$. (The derivative can be calculated without factoring first, using the product rule for derivatives. It gives $-\rho^{10} + (1-\rho)(10\rho^9) = 10\rho^9 - 11\rho^{10} = (10 - 11\rho)\rho^9$ as before.)

4. (a) The number of failures in a given day, X, has the binomial distribution with parameters $n = 1500$ and $p = 0.001$. Thus, $P\{X \geq 2\} = 1 - p_X(0) - p_X(1) = 1 - (0.999)^{1500} - 1500(0.001)(0.999)^{1499}$.

(b) The approximate distribution of X is Poisson with parameter λ given by $\lambda = np = 1500(0.001) = 1.5$. Hence $P\{X \geq 2\} = 1 - p_X(0) - p_X(1) \approx 1 - e^{-1.5} - (1.5)e^{-1.5} = 1 - (2.5)e^{-1.5}$.

(c) We need to identify k that maximizes $p(k) = \frac{e^{-1.7}(1.7)^k}{k!}$. Since $\frac{p(k)}{p(k-1)} = \frac{1.7}{k}$ we see $p(0) < p(1) > p(2) > p(3) > \cdots$ so $k = 1$ is the most likely number of server failures in a given day.

5. (a) Let H_3 denote the event of heads showing up on all three flips. Let F denote the event that the fair coin is chosen.

$$P(H_3) = P(H_3|F)P(F) + P(H_3|F^c)P(F^c) = \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right) + \left(\frac{3}{4}\right)^3 \left(\frac{1}{2}\right) = \frac{35}{128}.$$

(b) Let H_2 denote the event of heads showing up twice. Let F denote the event that the fair coin is chosen.

$$P(H_2) = P(H_2|F)P(F) + P(H_2|F^c)P(F^c) = 3\times\left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right) + 3\left(\frac{3}{4}\right)^2 \times\left(\frac{1}{4}\right)\left(\frac{1}{2}\right) = \frac{51}{128}.$$

$$P(F|H_2) = \frac{P(H_2|F)P(F)}{P(H_2)} = \frac{3 \times \left(\frac{1}{2}\right)^3 \times \left(\frac{1}{2}\right)}{\frac{51}{128}} = \frac{\frac{3}{16}}{\frac{51}{128}} = \frac{8}{17}.$$

6. (a) The likelihood ratio $\Lambda(k) = \frac{p_1(k)}{p_0(k)} = \frac{k^2}{10}$ and therefore the ML rule is to declare H_1 whenever $\Lambda(X) \geq 1$, or $X \geq 4$, or equivalently, $X = 4$.

(b) The MAP rule declares H_1 whenever $\Lambda(X) \geq \frac{\pi_0}{\pi_1} = \frac{1}{3}$, or equivalently, $X \geq 2$.