University of Illinois Fall 2014

ECE 313: Hour Exam I

Monday, March 3, 2014
7:00 p.m. — 8:15 p.m.
228 Nat. History (Sec. E/9am and C/10am) & 314 Altgeld (D/11am and F/1pm)

1. **[18 points]** The Sweet Dreams Cookie Store sells n types of cookies, kept in n separate jars evenly spaced around the edge of a round table. One night, two neighborhood cats, Tom & Jerry, each sneak into the store, choose a jar at random (all jars being equally likely), take a cookie, and leave.

   (a) (6 points) Find the number of ordered pairs of jars (i.e. (jar Tom chooses, jar Jerry chooses)) that Tom & Jerry can choose to take cookies from. Assume they can choose the same jar.

   (b) (6 points) Assume now that Tom comes in first and empties the jar he chooses, so Jerry chooses a different jar at random, again all possibilities being equally likely. Find the number of ordered pairs of jars that Tom & Jerry can choose.

   (c) (6 points) Under the assumption in part (b), find the probability that Tom and Jerry pick jars in such a way that there are three or fewer jars between the two they choose (going around the table in the direction giving the shortest distance between the two jars). Provide an answer for all values of n with $n \geq 2$.

2. **[30 points]** Suppose n balls are thrown into n bins independently and uniformly at random, where $n \geq 2$ is an integer.

   (a) (5 points) Let X be the number of balls in the first bin. Find the pmf of X, $p_X(k)$, for $0 \leq k \leq n$. Be as explicit as possible.

   (b) (5 points) Find the mean and variance of $2X + 1$.

   (c) (5 points) What is the distribution of X in the limit as $n \to \infty$? (i.e. a good approximation of the distribution of X for n large).

   (d) (5 points) Find the conditional probability the first bin has one ball given that exactly one ball falls into the first two bins.

   (e) (5 points) Let A denote the event that the first bin is empty. Let B denote the event that the last bin is empty. Are A and B mutually independent?

   (f) (5 points) What is the probability there are no empty bins?

3. **[12 points]** The two parts below are unrelated.

   (a) (6 points) Suppose the fraction of people in Tokyo in favor of a certain referendum will be estimated by a poll. A confidence interval based on the Chebychev bound will be used. Suppose the width of the confidence interval would be 0.1 for sample size $n = 300$ and some given confidence level. How many samples would be needed instead to yield a confidence interval that has only half the width, for the same level of confidence?

   (b) (6 points) Suppose X has the pmf $p(k) = (1 - \rho)\rho^k$ for $k \geq 0$, where ρ is a parameter with $0 < \rho < 1$. (This is *not* quite the same as a geometric distribution.) Find $\widehat{\rho}_{ML}(10)$, the maximum likelihood estimate of ρ for observation $X = 10$.

4. **[15 points]** A particular cloud computing facility has 1500 servers. In a given day, each server fails independently with probability 0.001.

   (a) (5 points) Using the binomial distribution, write a simple expression with three terms for the probability that two or more servers fail in a given day. (Hint: It is one minus the probability that zero or one servers fail.)

(b) (5 points) Use the Poisson distribution to give an approximate expression with three terms for the same probability as in part (a).

(c) (5 points) Suppose instead that the number of server failures in a given day has the Possion distribution with parameter $\lambda = 1.7$. Determine the most likely number of server failures in a given day. Show your work.

5. [**15 points**] Two coins, one fair and one coming up heads with probability $\frac{3}{4}$, are in a pocket. One coin is drawn from the pocket, with each possibility having equal probability. The coin drawn is flipped three times.

   (a) [**5 points**] What is the probability that heads show up on all three flips?

   (b) [**10 points**] Given that a total of two heads show up on the three flips, what is the probability that the coin is fair?

6. [**10 points**] Consider a hypothesis testing problem in which the pmfs of the observation X under hypotheses H_1 and H_0 are given, respectively, by

$$p_1(k) = \frac{k^3}{100} \text{ for } k = 1, 2, 3, 4,$$

and

$$p_0(k) = \frac{k}{10} \text{ for } k = 1, 2, 3, 4.$$

   (a) [**5 points**] Find the ML decision rule, expressing your answer in terms of X in the simplest possible way.

   (b) [**5 points**] Find the MAP decision rule, expressing your answer in terms of X in the simplest possible way, using priors $\pi_0 = \frac{1}{4}$ and $\pi_1 = \frac{3}{4}$.