University of Illinois **Spring 2013**

ECE 313: Problem Set 11: Problems and Solutions
Failure rate function, jointly random variables, independence

Due: Wednesday, April 10 at 6 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.9–4.4

1. **[Failure rate]**

   (a) Dr. Doofenschmirtz devised a dastardly deception dependent on defective deodorant. Each can of Doofenschmirtz Deodorant is sold with a three-week warranty. Unknown to the citizens of the tri-state area, Dr. Doofenschmirtz has carefully screened his deodorant so that it always lasts longer than three weeks. At the end of three weeks, the radio tower on the Doofenschmirtz building sends out random self-destruct signals so that each can of deodorant spectacularly self-destructs some time before the beginning of the fifth week. Thus the time at which any given can self-destructs is a random variable, T, distributed according to

   $$f_T(t) = \begin{cases} 0.5 & 3 < t < 5 \\ 0 & \text{otherwise} \end{cases}$$

   This policy causes the failure rate, $h(t)$, to grow in a wonderfully evil unbounded fashion, much to the delight of Dr. Doofenschmirtz's peers in the International Institute for Inventors of Evil (IIIE). As the detective in charge of the case, it is your job to find $h(t)$.
   Solution:

   $$h(t) = \frac{f_T(t)}{1 - F_T(t)} = \begin{cases} 0 & t < 3 \\ \frac{1}{5-t} & 3 \leq t \leq 5 \\ \text{undefined} & t > 5 \end{cases}$$

   (b) The failure rates of real-world products also grow in an unbounded fashion, but usually not as spectacularly as Doofenschmirtz Deodorant. For example, the failure rate of a typical real-world product might be

   $$h(t) = \begin{cases} t & t > 0 \\ 0 & t \leq 0 \end{cases}$$

   Find the pdf of random variable T, the time at which such a product fails.
   Solution:

   $$1 - F_T(t) = e^{-\int_{-\infty}^{t} h(u)du} = \begin{cases} 1 & t \leq 0 \\ e^{-\frac{t^2}{2}} & t \geq 0 \end{cases}$$

   $$f_T(t) = -\frac{\partial}{\partial t}(1 - F_T(t)) = \begin{cases} 0 & t \leq 0 \\ te^{-\frac{t^2}{2}} & t \geq 0 \end{cases}$$

2. **[Discrete random variables]**
   Two fair six-sided dice are rolled. One of the dice shows Z_1 pips, the other shows Z_2 pips. The random variables X and Y are defined as follows:

   $$X = \min(Z_1, Z_2)$$
   $$Y = \max(Z_1, Z_2)$$

(a) Sketch the support, in the (u, v) plane, of the joint pmf $p_{X,Y}(u, v)$.

Solution: The support is the set of all pairs of integers (u, v) such that $1 \le u \le v \le 6$.

(b) Find the joint pmf $p_{X,Y}(u, v)$.

Solution: For any particular $u \ne v$, there are two ways in which $X = u, Y = v$ can occur (either $Z_1 = u$ and $Z_2 = v$, or vice versa). If $u = v$, there is only one way in which $X = u, Y = v$ can occur. Thus

$$p_{X,Y}(u, v) = \begin{cases} \frac{1}{18} & 1 \le u < v \le 6 \\ \frac{1}{36} & 1 \le u = v \le 6 \\ 0 & \text{otherwise} \end{cases}$$

(c) Find the marginal pmf of Y.

Solution: There are $v - 1$ ways in which $X = u$, $Y = v$ and $u < v$ can occur. There is one way in which $X = Y = u = v$ can occur. Thus

$$p_Y(v) = \begin{cases} \frac{1}{18}(v-1) + \frac{1}{36} = \frac{2v-1}{36} & 1 \le v \le 6 \\ 0 & \text{otherwise} \end{cases}$$

It is also possible, and correct, to derive this directly from the problem statement without any use of the joint pmf, but it takes a few extra steps that way.

(d) Find $p_{X|Y}(u|v)$.

Solution:

$$p_{X|Y}(u|v) = \frac{p_{X,Y}(u, v)}{p_Y(v)} = \begin{cases} \frac{2}{2v-1} & 1 \le u < v \le 6 \\ \frac{1}{2v-1} & 1 \le u = v \le 6 \\ 0 & 1 \le v \le 6, v < u \text{ or } u < 1 \\ \text{undefined} & v < 1 \text{ or } v > 6 \end{cases}$$

(e) Find the joint CDF, $F_{X,Y}(c, d)$.

Solution: This problem is really best solved by looking at the graph of $p_{X,Y}(u, v)$. First, notice that if $c \ge d$, then we are adding all values of $p_{X,Y}(u, v)$ within the band $v \le d$, thus for example

$$F_{X,Y}(d + 1, d) = F_{X,Y}(d, d) = F_Y(d) = \sum_{v=1}^{d} \frac{2v-1}{36} = \frac{d(d+1) - d}{36} = \frac{d^2}{36}$$

where we have used the identity $\sum_{v=1}^{d} v = \frac{d(d+1)}{2}$. Second, notice that if $c < d$, then the set of (u, v) pairs being summed includes two regions: a triangular region whose sum is $F_Y(c)$, and a rectangular region containing $\min((6 - c) \times c, (d - c) \times c)$ pairs, each with probability $\frac{1}{18}$. Putting it all together, we have

$$F_{X,Y}(c, d) = \begin{cases} 1 & 6 < d, 6 < c \\ \frac{d^2}{36} & 1 \le d \le 6, d \le c \\ \frac{c^2 + \min(2c(d-c), 2c(12-c))}{36} = \frac{\min(12c, 2cd) - c^2}{36} & 1 \le c < 6, c < d \\ 0 & \text{otherwise} \end{cases}$$

(f) Find $E[Y-X]$.

Solution: The value $(Y-X)=(v-u)$ is a constant along every diagonal of the (u,v) plane. There are five points for which $(v-u)=1$, four for which $(v-u)=2$, and so on, thus

$$E[Y-X]=\sum_{u,v}(v-u)p_{X,Y}(u,v)=\frac{1\times 5}{18}+\frac{2\times 4}{18}+\frac{3\times 3}{18}+\frac{4\times 2}{18}+\frac{5\times 1}{18}=\frac{35}{18}$$

3. **[Continuous random variables]**
Consider a pair of continuous-valued random variables, X and Y, whose pdf is given by the following pyramid, for some constant height A:

$$f_{X,Y}(u,v)=\begin{cases} Au & 0\le u\le v\le 1\\ Av & 0\le v\le u\le 1\\ 0 & \text{otherwise}\end{cases}$$

(a) What is the value of the constant A?

Solution: The volume of a pyramid is $V=\frac{1}{3}bh$, where b is the area of the base, and h is the height. The base of the $f_{X,Y}(u,v)$ pyramid is $b=1$, and its height is $h=A$. The total volume of this pyramid must be $V=1$, therefore $A=3$.

(b) Find the marginal pdf of Y.

Solution:

$$f_Y(v)=\int_{-\infty}^{\infty}f_{X,Y}(u,v)du=\begin{cases}\int_0^v 3udu+\int_v^1 3vdu=3v-1.5v^2 & 0\le v\le 1\\ 0 & \text{otherwise}\end{cases}$$

(c) Find $f_{X|Y}(u|v)$.

Solution:

$$f_{X|Y}(u|v)=\frac{f_{X,Y}(u,v)}{f_Y(v)}=\begin{cases}\frac{u}{v-0.5v^2} & 0\le u\le v\le 1\\ \frac{1}{1-0.5v} & 0\le v\le u\le 1\\ 0 & u\notin[0,1],\ 0\le v\le 1\\ \text{undefined} & v\notin[0,1]\end{cases}$$

(d) Find the CDF, $F_{X,Y}(c,d)$.

Solution: $F_{X,Y}(c,d)=\int_{-\infty}^{c}\int_{-\infty}^{d}f_{X,Y}(u,v)dvdu$. Let's assume, first, that $0\le c\le d\le 1$. Then

$$\begin{aligned}F_{X,Y}(c,d) &= \int_0^c dv\int_0^v 3udu+\int_0^c du\int_0^u 3vdv+\int_c^d dv\int_0^c 3udu\\ &= \frac{3dc^2-c^3}{2},\quad 0\le c\le d\le 1\end{aligned}$$

Now if $d<c$, we just swap the positions of d and c. If $d>1$, then the third integral has an upper limit of 1, instead of d. If either c or d is below 0, then the CDF is zero. Putting it all together, we get

$$F_{X,Y}(c,d)=\frac{1}{2}\left(3\min(1,\max(c,d))\max(0,\min(c,d))^2-\max(0,\min(c,d))^2\right)$$

which can also be written

$$F_{X,Y}(c,d) = \begin{cases} 0 & c \le 0 \text{ or } d \le 0 \\ \frac{1}{2}\left(3dc^2 - c^3\right) & 0 \le c \le d \le 1 \\ \frac{1}{2}\left(3c^2 - c^3\right) & 0 \le c \le 1 \le d \\ \frac{1}{2}\left(3cd^2 - d^3\right) & 0 \le d \le c \le 1 \\ \frac{1}{2}\left(3d^2 - d^3\right) & 0 \le d \le 1 \le c \\ 1 & \text{otherwise} \end{cases}$$

(e) Find $E[X+Y]$.

Solution:

$$\begin{aligned} E[X+Y] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(u+v)f_{X,Y}(u,v)dudv \\ &= \int_0^1 dv\int_0^v 3u^2du + \int_0^1 dv\int_0^v 3uvdu \\ &\quad + \int_0^1 du\int_0^u 3v^2dv + \int_0^1 du\int_0^u 3uvdv \\ &= \frac{5}{4} \end{aligned}$$

This can also be done using $E[X+Y] = E[X] + E[Y]$, then solving for both $E[X]$ and $E[Y]$ using their marginal pdfs.

4. **[Uniform random variables]**
Consider a pair of continuous-valued random variables, X and Y, whose pdf is given by the following, for some constant value of A:

$$f_{X,Y}(u,v) = \begin{cases} \frac{1}{A} & 0 < u,v < 1 \\ \frac{1}{A} & 0.5 < v < 1.5, 1 < u < 2 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is the value of the constant A?

Solution: The total volume of the pdf $f_{X,Y}(u,v)$ is $V = 2A$. Since we must have $V = 1$, it follows that $A = 0.5$.

(b) Find the marginal pdf of Y.

Solution:

$$f_Y(v) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)du = \begin{cases} 0.5 & 0 < v < 0.5 \\ 1 & 0.5 < v < 1 \\ 0.5 & 1 < v < 1.5 \\ 0 & \text{otherwise} \end{cases}$$

(c) Find the marginal pdf of X.

Solution:

$$f_X(u) = \int_{-\infty}^{\infty} f_{X,Y}(u,v)dv = \begin{cases} 0.5 & 0 < u < 2 \\ 0 & \text{otherwise} \end{cases}$$

(d) Find $f_{X|Y}(u|v)$.
Solution:

$$f_{X|Y}(u|v) = \frac{f_{X,Y}(u,v)}{f_Y(v)} = \begin{cases} 1 & 0 < u < 1,\ 0 < v < 0.5 \\ 0.5 & 0 < u < 2,\ 0.5 < v < 1 \\ 1 & 1 < u < 2,\ 1 < v < 1.5 \\ \text{undefined} & v \notin [0, 1.5] \\ 0 & \text{otherwise} \end{cases}$$

(e) Find the CDF, $F_{X,Y}(c,d)$.
Solution:

$$F_{X,Y}(c,d) = \begin{cases} 0 & c \le 0 \text{ or } d \le 0 \\ 0.5cd & 0 \le c, d \le 1 \\ 0.5d & 0 \le d \le 0.5,\ 1 \le c \\ 0.5 + 0.5(c-1)(d-0.5) & 0.5 \le d \le 1.5,\ 1 \le c \le 2 \\ 0.5 + 0.5(d-0.5) & 0.5 \le d \le 1.5,\ 2 \le c \\ 0.5c & 0 \le c \le 1,\ 1 \le d \le 1.5 \\ 0.5c & 0 \le c \le 2,\ 1.5 \le d \\ 1 & 2 \le c,\ 1.5 \le d \end{cases}$$

(f) Find $E[X+Y]$.
Solution:

$$E[X+Y] = E[X] + E[Y] = 1 + 0.75 = 1.75$$

5. **[Independent Random Variables]**
Determine, for each of the following joint distributions, whether or not X and Y are independent random variables.

(a)

$$f_{X,Y}(u,v) = \frac{\lambda^2}{4} e^{-\lambda|u|} e^{-\lambda|v|}, \quad -\infty < u < \infty,\ -\infty < v < \infty$$

Solution: Independent. This pdf can be factored into a term dependent on u, and a term dependent on v.

(b)

$$f_{X,Y}(u,v) = \frac{\lambda^2}{2} e^{-\lambda|u+v|} e^{-\lambda|u-v|}, \quad -\infty < u < \infty,\ -\infty < v < \infty$$

Solution: Not Independent. There are a few ways to see this:
(1) $f_{X,Y}(u,v) \neq f_X(u) f_Y(v)$ for any possible choice of $f_X(u)$ and $f_Y(v)$. This can be proven by noticing that $f_{X,Y}(0,0) f_{X,Y}(1,1) \neq f_{X,Y}(0,1) f_{X,Y}(1,0)$.
(2) $f_{Y|X}(v|u) \neq f_Y(v)$. This can be proven without explicitly calculating the marginal and conditional pdfs if one realizes that, for any particular value of u, $f_{Y|X}(v|u) = f_{X,Y}(u,v)/f_X(u)$, therefore as a function of v, $f_{Y|X}(v|u) \propto f_{X,Y}(u,v)$. Notice that $f_{X,Y}(0,v)$ has the shape $0.5\lambda^2 e^{-2\lambda|v|}$, but $f_{X,Y}(1,v)$ is constant in the range $-1 \le v \le 1$. Since $f_{X,Y}(u,v)$ as a function of v has different shapes at $u = 0$ and $u = 1$, it follows necessarily that $f_{Y|X}(v|u)$ has different shapes at $u = 0$ and $u = 1$, therefore $f_{Y|X}(v|u) \neq f_Y(v)$.