University of Illinois **Spring 2013**

ECE 313: Problem Set 11
Failure rate function, jointly random variables, independence

Due: Wednesday, April 10 at 6 p.m.
Reading: *ECE 313 Course Notes,* Sections 3.9–4.4

1. **[Failure rate]**

   (a) Dr. Doofenschmirtz devised a dastardly deception dependent on defective deodorant. Each can of Doofenschmirtz Deodorant is sold with a three-week warranty. Unknown to the citizens of the tri-state area, Dr. Doofenschmirtz has carefully screened his deodorant so that it always lasts longer than three weeks. At the end of three weeks, the radio tower on the Doofenschmirtz building sends out random self-destruct signals so that each can of deodorant spectacularly self-destructs some time before the beginning of the fifth week. Thus the time at which any given can self-destructs is a random variable, T, distributed according to

   $$f_T(t) = \begin{cases} 0.5 & 3 < t < 5 \\ 0 & \text{otherwise} \end{cases}$$

   This policy causes the failure rate, $h(t)$, to grow in a wonderfully evil unbounded fashion, much to the delight of Dr. Doofenschmirtz's peers in the International Institute for Inventors of Evil (IIIE). As the detective in charge of the case, it is your job to find $h(t)$.

   (b) The failure rates of real-world products also grow in an unbounded fashion, but usually not as spectacularly as Doofenschmirtz Deodorant. For example, the failure rate of a typical real-world product might be

   $$h(t) = \begin{cases} t & t > 0 \\ 0 & t \leq 0 \end{cases}$$

   Find the pdf of random variable T, the time at which such a product fails.

2. **[Discrete random variables]**
   Two fair six-sided dice are rolled. One of the dice shows Z_1 pips, the other shows Z_2 pips. The random variables X and Y are defined as follows:

   $$\begin{aligned} X &= \min(Z_1, Z_2) \\ Y &= \max(Z_1, Z_2) \end{aligned}$$

   (a) Sketch the support, in the (u, v) plane, of the joint pmf $p_{X,Y}(u, v)$.

   (b) Find the joint pmf $p_{X,Y}(u, v)$.

   (c) Find the marginal pmf of Y.

   (d) Find $p_{X|Y}(u|v)$.

   (e) Find the joint CDF, $F_{X,Y}(c, d)$.

   (f) Find $E[Y - X]$.

3. **[Continuous random variables]**
   Consider a pair of continuous-valued random variables, X and Y, whose pdf is given by the following pyramid, for some constant height A:

$$f_{X,Y}(u,v) = \begin{cases} Au & 0 \le u \le v \le 1 \\ Av & 0 \le v \le u \le 1 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the value of the constant A?
   (b) Find the marginal pdf of Y.
   (c) Find $f_{X|Y}(u|v)$.
   (d) Find the CDF, $F_{X,Y}(c,d)$.
   (e) Find $E[X+Y]$.

4. **[Uniform random variables]**
   Consider a pair of continuous-valued random variables, X and Y, whose pdf is given by the following, for some constant value of A:

$$f_{X,Y}(u,v) = \begin{cases} \frac{1}{A} & 0 < u, v < 1 \\ \frac{1}{A} & 0.5 < v < 1.5, 1 < u < 2 \\ 0 & \text{otherwise} \end{cases}$$

   (a) What is the value of the constant A?
   (b) Find the marginal pdf of Y.
   (c) Find the marginal pdf of X.
   (d) Find $f_{X|Y}(u|v)$.
   (e) Find the CDF, $F_{X,Y}(c,d)$.
   (f) Find $E[X+Y]$.

5. **[Independent Random Variables]**
   Determine, for each of the following joint distributions, whether or not X and Y are independent random variables.

   (a)

$$f_{X,Y}(u,v) = \frac{\lambda^2}{4} e^{-\lambda|u|} e^{-\lambda|v|}, \quad -\infty < u < \infty, \ -\infty < v < \infty$$

   (b)

$$f_{X,Y}(u,v) = \frac{\lambda^2}{2} e^{-\lambda|u+v|} e^{-\lambda|u-v|}, \quad -\infty < u < \infty, \ -\infty < v < \infty$$