University of Illinois **Spring 2013**

ECE 313: Problem Set 10
Joint Distributions, Independence

Due: Wednesday, November 2 at 4 p.m.
Reading: *ECE 313 Course Notes,* Sections 4.1-4.4.

1. **[A joint pmf]**
The joint pmf $p_{\mathbb{X},\mathbb{Y}}(u, v)$ of X and Y is shown in the table below.

	u=0	u=1	u=2	u=3
v=4	0	0.1	0.1	0.2
v=5	0.2	0	0	0
v=6	0	0.2	0.1	0.1

   (a) Find the marginal pmfs $p_X(u)$ and $p_Y(v)$.

   (b) Let $Z = X + Y$. Find p_Z, the pmf of Z.

   (c) Are X and Y independent random variables? Justify your answer.

   (d) Find $p_{Y|X}(v|3)$ for all v and find $E[Y|X = 3]$.

2. []
Consider the following binary hypothesis testing problem. If hypothesis H_0 is true, the continuous random variable $\mathbb{X}$ is uniformly distributed on the interval $(-2, 2)$, while if hypothesis H_1 is true, the pdf of $\mathbb{X}$ is $f_1(u) = \begin{cases} \frac{1}{4}(2 - |u|), & |u| < 2, \\ 0, & \text{otherwise.} \end{cases}$

   (a) The *maximum-likelihood* decision rule can be stated in the form $|\mathbb{X}| \underset{\mathsf{H}_{1-x}}{\overset{\mathsf{H}_x}{\gtrless}} \eta$. Specify whether x denotes 0 or 1, and find the values of η, the probability of false alarm P_{FA}, and the probability of missed detection P_{MD}.

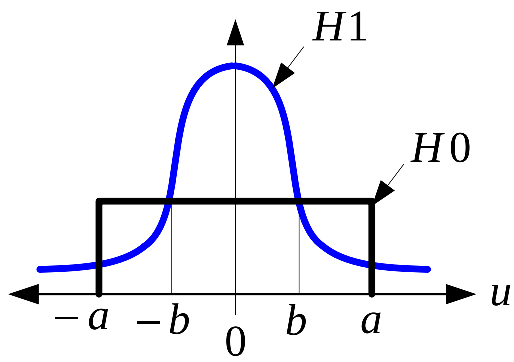
   (b) Suppose the hypotheses have *a priori* probabilities $\pi_0 = 1/3$ and $\pi_1 = 2/3$. What is the error probability $P(E)$ of the maximum-likelihood decision rule?

   (c) The MAP (also known as the minimum-error-probability or Bayesian) decision rule can be stated in the form $|\mathbb{X}| \underset{\mathsf{H}_{1-x}}{\overset{\mathsf{H}_x}{\gtrless}} \xi$. Specify whether x denotes 0 or 1, and find the values of ξ and the error probability $P(E)$.

3. []
An observation X is drawn from a standard normal distribution (i.e. $N(0, 1)$) if hypothesis H_1 is true and from a uniform distribution with support $[-a, a]$ if hypothesis H_0 is true. As shown in the figure below (under part (b)), the pdfs of the two distributions are equal when $|u| = b$.

   (a) Express the maximum likelihood (ML) decision rule in a simple way, in terms of the observation X and the constants a and b.

   (b) Shade and label the regions in the figure below such that the area of one of the regions is $p_{false\ alarm}$ and the area of the other region is p_{miss}.



(c) Express $p_{false\ alarm}$ and p_{miss} for the ML decision rule in terms of the constants a, b, and the Φ function or Q function with positive arguments.

(d) Determine the maximum *á posteriori* probability (MAP) rule when $a = \frac{3}{2}$, $b = 0.6$, and the probability of hypothesis $H1$ being true is. $\pi_1 = \frac{\sqrt{2\pi}}{3+\sqrt{2\pi}}$.

4. []
Consider the following binary hypothesis testing problem. If hypothesis H_0 is true, the continuous random variable $\mathbb{X}$ is uniformly distributed on $(-1, 1)$, while if hypothesis H_1 is true, the pdf of $\mathbb{X}$ is
$$f_1(u) = \begin{cases} C(1-|u|), & |u| < 1, \\ 0, & \text{otherwise.} \end{cases}$$

(a) Find the value of C.

(b) Find the decision region Γ_0 for the *maximum-likelihood* decision rule.
Remember that Γ_0 is the set of all real numbers such that if $\mathbb{X} \in \Gamma_0$, the decision is that H_0 is the true hypothesis.