University of Illinois Spring 2013

ECE 313: Final Exam

Friday May 3, 2013
8:00 a.m. — 11:00 a.m.

1. (a) This problem is solved by counting. The number of ways to picking up x Golden Eagles in a total of k coins is calculated as follows. We first select x American Eagle coins and then select $k-x$ Maple Leafs: The number of ways to select x Golden Eagles in the first step is given by $\binom{g}{x}$, and the number of ways to select $k-x$ Maple Leafs is given by $\binom{m}{k-x}$ and so there are $\binom{g}{x} \cdot \binom{m}{k-x}$ ways of selecting a total of k coins with x American Eagles. We normalize by the total number of ways to catch k fish, to get the solution: $\frac{\binom{g}{x} \cdot \binom{m}{k-x}}{\binom{g+m}{k}}$.

   (b) We first select the three coins to be selected again from our original picking, and then we select the $n-3$ other coins from the $g+m-x$ coins that were not Golden Eagles selected the first time. Therefore we have $\binom{x}{3}\binom{g+m-x}{n-3}$ total number of ways, giving us the probability of $\frac{\binom{x}{3}\binom{g+m-x}{n-3}}{\binom{g+m}{n}}$.

2. (a) FALSE. For instance A and B could be independent, and in that case the quantity on the right should be $2P(A)$.
   $P(A|B)P(B) + P(A^c|B)P(B) = P(A)$ is a FALSE statement.
   TRUE. Adding $P(A \cap B \cap C)$ to the RHS would make for an equality.

   (b) FALSE. These are conditional probabilities and could both be zero.
   FALSE. This is the average probability of error, which could be zero.
   FALSE. It could be that the average error probability is zero for both ML and MAP decision rules, but the prior probabilities are different.
   TRUE. When the prior probabilities are equal, the MAP and ML rule decision rules coincide.

3. (a) Using the law of total probability, the solution is $\sum_{k=1}^{\infty} \frac{1}{3^k} \frac{1}{10} \frac{9^{k-1}}{10^{k-1}} = \frac{1}{21}$.

   (b) Using Bayes rule, the solution is: $\frac{\frac{1}{3^2}\frac{1}{10}\frac{9}{10}}{\frac{1}{21}} = \frac{21}{100}$.

   (c) TRUE. The more the questions asked the larger the chance of answering more questions correctly.

   (d) $\mathbb{X} \leq \mathbb{Y}$, since the number of questions correctly is no more than the total number of questions asked in the show. Since $E[\mathbb{Y}] = 10$, the solution follows by the linearity of expectation.

4. (a) Using the Laplace-DeMoivre approximation, $P(N \leq 580)$ is well approximated by $P(Z \leq 580.5)$ where Z is Gaussian with mean 600 and variance 400. Normalizing to standard notation this latter probability is $Q\left(\frac{19.5}{20}\right)$.

(b) We first find the support of $\mathbb{X}$ and $\mathbb{Y}$ to be between $[-6, 6]$. By the convolution formula, the pdf of $\mathbb{Z}$ is an isosceles triangle with the corner points (-12,0), (12,0) and (1/6,0).

(c) The support set is a diamond with area of 2. So we get $C = 0.5$.

(d) No, $\mathbb{X}$ and $\mathbb{Y}$ are not independent – their support set is not rectangular in shape.

(e) $\mathbb{X}$ and $\mathbb{Y}$ are uncorrelated, because the pdf is symmetric with respect to the midpoint $X = Y = 0$.

5. (a) $Cov(X, Z) = Cov(X, X + Y) = Var(X) = 7$.
$Var(Z) = Var(X + Y) = Var(X) + Var(Y) = 16$.
$E(Z) = E(X + Y) = 3 + 0 = 3$.

$$\begin{aligned} L^*(Z) &= \mu_X + \frac{Cov(X,Z)}{Var(Z)}(Z - E(Z)) \\ &= 3 + \frac{7}{16}(Z-3) = 3 + \frac{7(Z-3)}{16}, \end{aligned}$$

Hence, $\boxed{L^*(Z) = 3 + \frac{7(Z-3)}{16}.}$

(b) $Cov(U, Z) = Cov(X - Y, X + Y) = Var(X) - Var(Y) = -2$.
$Var(Z) = Var(X+Y) = Var(X)+Var(Y) = 16$. $Var(U) = Var(X-Y) = Var(X)+Var(Y) = 16$. So $\rho_{U,Z} = \frac{-1}{8}$.

(c) $L^*(X) = \mu_X + \frac{Cov(X,V)}{Var(V)}(V - \mu_V)$. So it must be that $c = \mu_X = 3$ and $Cov(X, V) = 0$.

6. (a) Since A and B are independent, $P(AB) = P(A)P(B)$, therefore $P(B) = 1/3$, $P(B^c) = 2/3$, and $P(A^cB^c) = P(A^c)P(B^c) = 1/3$.

(b) $P(AC^C) = P(A) - P(AC) = \frac{1}{3}$. $P(A^CC^C) = P(C^C) - P(AC^C) = \frac{1}{6}$.

(c) $\frac{2}{3} = \frac{P(AD)}{P(D)} \le \frac{P(A)}{P(D)}$, so $P(D) \le \frac{3}{2}P(A) = \frac{3}{4}$. There is no minimum value of $P(D)$, except that if $P(D) = 0$, then $P(A|D)$ would be undefined, therefore $P(D) > 0$.

(d) In order to have \$182 left at the end of ten games, A must occur at least nine times. Define X to be the number of occurrences of A; then

$$\Pr\{100 + \text{Winnings} \ge 182\} = p_X(9) + p_X(10) = \left(\binom{10}{9} + \binom{10}{10}\right)\left(\frac{1}{2}\right)^{10} = \frac{11}{1024}$$

7. (a)

$$0 = \frac{\partial}{\partial\lambda} f_T(t)\bigg|_{\hat\lambda} = \left(\frac{4}{\hat\lambda} - T\right)\frac{\hat\lambda^4 t^3 e^{-\hat\lambda t}}{3!}$$

Therefore $\hat\lambda = \frac{4}{T}$.

(b) The number of calls in 10 minutes is a Poisson random variable X, thus

$$\Pr\{X \le 4\} = \sum_{k=0}^{3}\frac{(10\lambda)^k e^{-10\lambda}}{k!} = e^{-10\lambda}\left(1 + 10\lambda + \frac{(10\lambda)^2}{2} + \frac{(10\lambda)^3}{6}\right) = \frac{19}{3e^2}$$

(c)

$$f_{X,Y}(u,v) = \begin{cases} (0.2)^2 e^{-0.2v} & 0 \le u \le v \\ 0 & \text{otherwise} \end{cases}$$

8. (a)

$$f_X(u) = \begin{cases} \frac{3}{7} & 0 \le u < 1 \\ \frac{2}{7} & 1 \le u \le 2 \\ \frac{3}{7} & 2 < u \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$f_{Y|X}(v|\frac{1}{2}) = \begin{cases} \frac{1}{2} & 0 \le v \le 1 \text{ OR } 2 < v \le 3 \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$F_{X,Y}(u, \frac{1}{2}) = \begin{cases} 0 & u < 0 \\ \frac{u}{14} & 0 \le u \le 3 \\ \frac{3}{14} & u \ge 3 \end{cases}$$

9. (a)

$$f_P(w) = f_M(w) * f_B(w) = \begin{cases} \frac{1}{2}(w-1) & 2 \le w \le 3 \\ \frac{1}{2} & 3 \le w \le 4 \\ \frac{1}{2}(5-w) & 4 \le w \le 5 \\ 0 & \text{otherwise} \end{cases}$$

(b)

$$\begin{aligned} \text{Var}(P) &= \text{Var}(M) + \text{Var}(B) + 2\text{Cov}(M, B) \\ &= \text{Var}(M) + \text{Var}(B) + 2\rho_{MB}\sqrt{\text{Var}(B)\text{Var}(M)} \\ &= \frac{1}{3} + \frac{1}{12} - \frac{2}{2}\sqrt{\frac{1}{36}} \\ &= \frac{1}{4} \end{aligned}$$

(c)

$$\begin{aligned} f_Z(0) &= \int_{-1}^{1} du f_X(u) f_{Y|X}(0-u|u) \\ &= \int_{-1}^{1} du \frac{1-u^2}{4} \\ &= \frac{1}{4}\left[u - \frac{u^3}{3}\right]_{-1}^{1} \\ &= \frac{1}{3} \end{aligned}$$

10. (a)

$$1 - (p_X(0) + p_X(1)) = 1 - \binom{4}{0}\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4 + \binom{4}{1}\left(\frac{1}{3}\right)^1\left(\frac{2}{3}\right)^3 = 1 - \frac{48}{81} = \frac{33}{81}$$

(b)

$$p_T(t) = \begin{cases} (t-1)\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^{t-2} & t \ge 2 \\ 0 & \text{otherwise} \end{cases}$$

(c)

$$\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{8}{243}$$

(d)

$$p_{T|F}(t|f) = \begin{cases} \left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{t-f-1} & t \geq f+1 \\ 0 & \text{otherwise} \end{cases}$$