University of Illinois Spring 2013

ECE 313: Hour Exam II

Monday April 15, 2013
7:00 p.m. — 8:15 p.m.

1. **[25 points]** Alice is fishing. Fish bite randomly according to a Poisson prcess, with $\lambda_A = 3$ bites/hour. Define the following two random variables:

$$\begin{aligned} X &= \text{the total number of fish that Alice catches in the first TWO hours of fishing} \\ T &= \text{the time that Alice waits until the first fish bites.} \end{aligned}$$

   (a) **[4 points]** What is $E[X]$?

   (b) **[7 points]** What is $\Pr\{T > 1\}$, the probability that Alice waits more than one hour for the first fish?

   (c) **[7 points]** Alice begins fishing at noon. When Bob arrives at 1:00, she has still caught no fish. What is $E[T|T > 1]$?

   (d) **[7 points]** Fish bite Bob's fishing hook according to a Poisson process with $\lambda_B = 5$ fish/hour. Alice and Bob fish in different parts of the stream, thus the number of fish biting Alice's hook is independent from the number of fish biting Bob's hook. Let Y be the total number of fish caught by both Alice and Bob in one hour. What is $p_Y(2)$?

2. **[25 points]** Let $\mathbb{X}$ be a continuous random variable with variance 100. The pdf of $\mathbb{X}$ is symmetric around the origin, i.e., for all u, we have $f_{\mathbb{X}}(u) = f_{\mathbb{X}}(-u)$.

   (a) **[5 points]** Find the mean of $\mathbb{X}$.

   (b) **[5 points]** Let $\mathbb{Y} = 3\mathbb{X} + 4$. Find the mean of $\mathbb{Y}^2$.

   (c) **[5 points]**Let $\mathbb{W} = |\mathbb{X}|$. Then it is claimed that the pdf of $\mathbb{W}$, $f_{\mathbb{W}}(u) = 2f_{\mathbb{X}}(u)$ for all $u > 0$. Mark TRUE or FALSE by checking one box below. You need not justify your answer.

TRUE	FALSE
☐	☐

   (d) **[5 points]** Suppose further that $\mathbb{X}$ is Gaussian. Express the value of the probability $P[|\mathbb{X}| \leq 10]$ in terms of the $Q(\cdot)$ function.

   (e) **[5 points]** Continue with the supposition that $\mathbb{X}$ is Gaussian. Now 100 independent tosses of a fair coin are made. Let $\mathbb{Z}$ be the number of Heads that show up. Which of the following probabilities most closely approximates the probability $P[\mathbb{Z} \leq 40]$? You need not justify your answer.

$P[\mathbb{X} \geq 17]$	$P[\mathbb{X} \geq 18]$	$P[\mathbb{X} \geq 19]$	$P[\mathbb{X} \geq 16]$
☐	☐	☐	☐

3. **[25 points]** Consider the following binary hypothesis testing situation: if H_0 is true then a random variable $\mathbb{X}$ is uniformly distributed from $[0, 2]$. If an alternate hypothesis H_1 is true then $\mathbb{X}$ is uniformly distributed from $[1, 5]$.

   (a) **[5 points]** Find the decision region Γ_0 for the maximum-likelihood decision rule. Remember that Γ_0 is the set of all real numbers such that if $\mathbb{X} \in \Gamma_0$, the decision is that H_0 is the true hypothesis.

   (b) **[5 points]** Find the probability of missed detection P_{MD} under the maximum-likelihood decision rule.

   (c) **[10 points]** Suppose we need the probability of missed detection to be zero. Under this condition, find the decision region Γ_0 that yields the *smallest* value of probability of false alarm. Find the corresponding smallest value of probability of false alarm $P_{\text{FA}}^{\min}$.

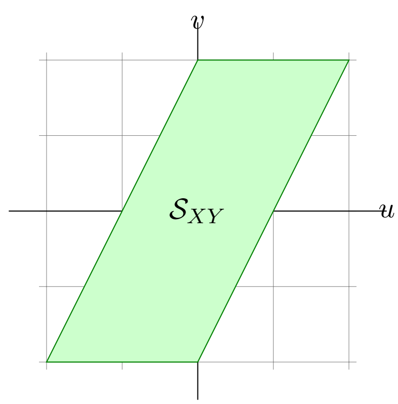
   (d) **[5 points]** This problem part is unrelated to the above 3 parts. Suppose π_0 and π_1 (with $\pi_0 < \pi_1$) are the prior probabilities of the binary hypotheses H_0 and H_1 respectively. Unfortunately there is no data observed, but we still need to decide to pick one of the two hypotheses. Consider the following decision rule. We independently toss a biased coin, with probability of Heads equal to p, and pick H_0 if Heads appears and H_1 otherwise. For what value of bias p does the decision rule yield the *smallest* average probability of error P_e? You do not need to justify your answer.

$p = 0$	$p = 1$	$p = \frac{1}{2}$	$p = \pi_0$
□	□	□	□

4. **[25 points]** Random variables X and Y are distributed as

$$f_{X,Y}(u, v) = \begin{cases} \frac{1}{8} & (u, v) \in \mathcal{S}_{XY} \\ 0 & \text{otherwise} \end{cases}$$

   where $\mathcal{S}_{XY}$ is the quadrilateral region whose corners are at $(-2, -2)$, $(0, 2)$, $(2, 2)$, and $(0, -2)$, as shown here:



   (a) **[6 points]** Are X and Y independent? Tell us how you know.

   (b) **[6 points]** Sketch $f_{X|Y}(u|1)$. Label the minimum and maximum values on both axes.

   (c) **[7 points]** Sketch $f_X(u)$. Label the minimum and maximum values on both axes.

   (d) **[6 points]** The CDF $F_{X,Y}(c, d)$ for the particular point $(c, d) = (0, 1)$ is computed by integrating the pdf over some region. Sketch the region that must be integrated in order to find $F_{X,Y}(0, 1)$, then integrate to find the numerical value of $F_{X,Y}(0, 1)$.