University of Illinois **Spring 2012**

ECE 313: Problem Set 3
Conditional probability, Bernoulli, and Binomial PMFs

Due: Wednesday, February 8 at 4 p.m.
Reading: *ECE 313 Course Notes,* Sections 2.3–2.4.

1. **[Alice, Bob, and Chess]**
   Busses are often late. Bus 5A Amber and bus 9B Brown are late with probabilities $P(A) = 0.2$ and $P(B) = 0.3$, respectively. A reporter from the Daily Illini has observed that, whenever bus 5A is late, bus 9B is also late, i.e., $P(B|A) = 1.0$. The reporter suspects that Bob, the driver of bus 9B, and Alice, the driver of bus 5A, are secretly meeting to play chess once every few days, and that their chess games make them late.

   (a) Alice defends herself by saying that $P(AB)$ is only $\frac{1}{5}$; since $P(AB) = P(A)$, she argues, the events A and B must be independent. What's wrong with her argument?

   (b) Under these circumstances (including the assumption $P(AB) = 1/5$ from part (a)), it might be reasonable to evaluate Bob's reliability by the probability $P(B^c|A^c)$. Find $P(B^c|A^c)$.

   (c) The reporter wants to find out how often Bob and Alice play chess. Let C be the event "Bob and Alice play chess." The reporter postulates that $P(A|C) = 1$ and $P(B|C) = 1$, i.e., a chess game always makes them late. She further postulates that, on days without chess, Bob and Alice are independent, i.e., $P(AB|C^c) = P(A|C^c)P(B|C^c)$. Under these assumptions, what is $P(C)$?

2. **[Independent and dependent events in a Karnaugh map]**
   Mad scientist Sheila Frankenstein has created a monster. Let A be the event "Monster eats roses," and let B be the event "Monster reads Shakespeare."

   (a) Suppose $P(A) = 1/3$, $P(B) = 1/4$, and the events A and B are independent. Draw a Karnaugh map of the experimental outcomes. Fill in the probabilities $P(AB)$, $P(AB^c)$, $P(A^cB)$, and $P(A^cB^c)$.

   (b) Suppose $P(A) = 1/3$, $P(B) = 1/4$, and $P(B|A) = 1/8$. Draw a Karnaugh map of the experimental outcomes. Fill in the probabilities $P(AB)$, $P(AB^c)$, $P(A^cB)$, and $P(A^cB^c)$.

3. **[Bernoulli Bets and Binomial Bets]**
   Your uncle, Ray, is an adventurous soul who never took ECE 313. He hears of a special offer in Las Vegas. The Spendthrift Casino on Metacalle Street is offering lottery tickets. Each ticket wins with a probability of $p = \frac{1}{50}$. A ticket can be purchased for any desired initial investment of C dollars. If the ticket wins, the investor receives $X = 100C$ dollars. If the ticket loses, the investor receives $X = 0$ dollars.

   (a) Ray wants to spend his life savings ($100) to buy one lottery ticket for $C = 100$ dollars. Let X be the amount of money he wins.

      i. Draw the PMF of X, $p_X(u)$ as a function of u.

      ii. What is $E[X]$?

iii. What is σ_X?

(b) Being a smart engineer who has taken ECE 313, you counsel a different strategy. You tell Ray to purchase $n = 100$ lottery tickets for $C = 1$ dollar each. Let Y be the amount of money Ray earns following your strategy.

i. Sketch the PMF of Y, $p_Y(v)$ as a function of v. Label the minimum, maximum, and modal values of v.

ii. What is $E[Y]$?

iii. What is σ_Y?

4. **[The Bizarre Bernoulli]**
The Bernoulli distribution $p_X(u)$ is one of the only interesting probability distributions that is strictly bounded, on at least one side, by a one-standard-deviation bound, in the following sense. Recall that the Bernoulli distribution is defined by:

$$p_X(u) = \begin{cases} p & u = 1 \\ 1-p & u = 0 \\ 0 & \text{otherwise} \end{cases}$$

Prove the following statement: for every value of p, either $E[X] - \sigma_x \leq 0$, or $E[X] + \sigma_x \geq 1$, or both, depending on the value of p.

(Note: this statement is interesting and worth proving because it implies that either $P\{X \geq E[X] - \sigma_X\} = 1$ or $P\{X \leq E[X] + \sigma_X\} = 1$ or both, depending on the value of p. There are not many types of random variables that can be so tightly bounded.)

5. **[Binomial Passenger Trains]**
Suppose that Amtrak constructs northbound trains using the following algorithm. The trainyard in Memphis has six train cars. If a car works, it is added to the northbound train; if not, it is left in Memphis. Each car works with a probability of $p = \frac{2}{3}$.

(a) What is the expected length, in cars, of a northbound train?

(b) What is the probability of a length-zero train?

(c) The number of passengers who can ride on a northbound train is $50X$, where X is the number of train cars. The number of passengers who *want* to ride the train, Y, is a uniformly distributed random variable, $p_Y(v) = \frac{1}{100}$ for $101 \leq v \leq 200$. What is the probability that Amtrak will be able to carry all of the passengers who wish to ride?

6. **[Adding Random Variables]**
The outcome of some particular experiment can be described by independent random variables, X and Y, distributed as

$$p_X(i) = \begin{cases} \binom{M}{i} p^i (1-p)^{M-i} & 0 \leq i \leq M \\ 0 & \text{otherwise} \end{cases}$$

and

$$p_Y(j) = \begin{cases} \binom{N}{j} p^j (1-p)^{N-j} & 0 \leq j \leq N \\ 0 & \text{otherwise} \end{cases}$$

This problem derives a few interesting facts about the random variable $Z = X + Y$, whose probability mass function is given by $p_Z(k)$.

(a) What is the probability $p_Z(0)$? Hint: in order for $Z = 0$ to occur, what must be the values of X and Y?

(b) What is the probability $p_Z(1)$? Hint: there are two different ways in which the event $Z = 1$ might occur.

(c) Generalizing from parts (a) and (b), write the probability mass function $p_Z(k)$ as an explicit summation over all the different values of i for which $k = i + j$ is possible.

(d) There is a different way to think about $p_Z(k)$. Remember that X can be considered to be the number of heads resulting from M coin tosses, for which the probability of heads resulting from any given toss is p. Similarly, Y is the number of heads resulting from N coin tosses. Z is therefore the number of heads resulting from $M + N$ coin tosses. Based on this interpretation of the experiment, write a form of $p_Z(k)$ that includes no explicit summation.

7. [**Computer Problem: Binomial random variables**]
An instance, i, of a Binomial random variable, X, can be generated in Matlab as follows. First, define variables containing your parameters, e.g., `p=0.7; N=100;`. Second, generate a random vector, q, containing N numbers, each of which is uniformly distributed between zero and one, using the command `q=rand(1,N);`. Third, find out which of the random numbers are smaller than p, using the command `b=(q<p);`. Finally, add up the number of ones: `i=sum(b);`.

(a) Notice that each element of the vector b is, itself, a Bernoulli random variable. Create a histogram of the values of the elements of b, using the `hist` function. Your histogram should show two possible values, $b = 0$ and $b = 1$, with probabilities of $1 - p$ and p respectively.

(b) Create 100 examples, i, of the binomial random variable, and plot a histogram. Your histogram should show a range of possible values between 0 and N, with a mode at $i = pN$.