**University of Illinois** Spring 2012

# ECE 313, Section F: Hour Exam I

Monday March 5, 2012
7:00 p.m. — 8:15 p.m.
163 Everitt

**Name:** **(in BLOCK CAPITALS)** _____

**University ID Number:** _____

**Signature:** _____

## Instructions

This exam is closed book and closed notes except that one 8.5"×11" sheet of notes is permitted: both sides may be used. Calculators, laptop computers, PDAs, iPods, cellphones, e-mail pagers, headphones, etc. are not allowed.

The exam consists of five problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but DO NOT convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75). Standard irrational numbers should be written in their standard form, e.g., write $e$, $\pi$, and $\sqrt{2}$, not 2.72, 3.14, and 1.41.

SHOW YOUR WORK; BOX YOUR ANSWERS. Answers without appropriate justification will receive very little credit. If you need extra space, use the back of the previous page. Draw a small box around each of your final numerical answers.

| Grading | |
|---|---|
| 1. 25 points | _____ |
| 2. 10 points | _____ |
| 3. 15 points | _____ |
| 4. 25 points | _____ |
| 5. 25 points | _____ |
| Total (100 points) | _____ |

1. **[25 points]** The average temperature in Urbana in March, $X$, is a random variable distributed according to the following pmf:

$$p_X(k) = \begin{cases} \frac{1}{20}\,|k-10| & 10-b \le k \le 10+b \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

   (a) **[6 points]** What is $b$?

   (b) **[7 points]** The monthly cost of heating Everitt Lab, in dollars, is $Y = 1000(25-X)^2$. You may assume that $E[X] = 10$ and $\mathrm{Var}(X) = 10$. What is $E[Y]$?

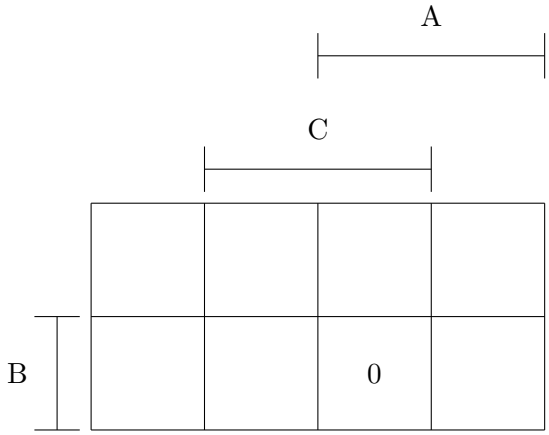
   (c) **[6 points]** Your cell phone reports the average temperature only in five-degree increments, but you are writing a report for the student senate that requires you to guess the temperature exactly. What is $P\{10 \le X \le 12 \,|\, 10 \le X \le 14\}$?

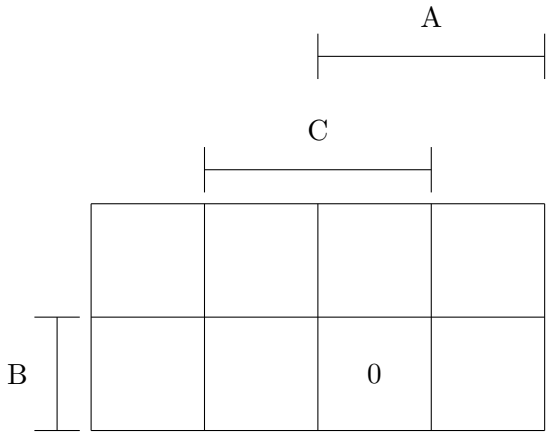
(d) **[6 points]** Define the following events:

- $A$: The average temperature is $X \leq 10$.
- $B$: There are at least 12 cloudy days in the month.

$A$ and $B$ are independent events, $P(B) = 0.3$, and $P(A)$ is as specified in Eq. 1. What is $P(A^c B^c)$?

2. **[10 points]** Suppose $A, B$, and $C$ are events for a probability experiment such that $P(A) = P(B) = P(C) = 2/3$, and $P(ABC) = 0$. Fill in the probabilities of all events indicated in the Karnaugh map provided. Show your work. An extra copy is provided for you to show your final answer. Circle your final answer.



3. **[15 points]** A particular sandwich bar offers customers their choice of ingredients. Every customer must choose exactly three different ingredients for his or her sandwich. There are five ingredients available: {Avocado, Broccoli, Cheese, Duck, and Egg}. If two sandwiches contain exactly the same ingredients, we say that they are the "same sandwich;" if at least one ingredient differentiates two sandwiches, we say that they are "different sandwiches."

   (a) **[5 points]** How many different sandwiches can be made that include Duck as one of the ingredients?

   (b) **[5 points]** Three customers in a row choose sandwiches. Each customer chooses three different ingredients uniformly at random from the set of five possible ingredients. What is the probability that two customers choose the same sandwich, but the third customer chooses a different sandwich?

   (c) **[5 points]** Alice's sandwich is different from Bob's (at least one of the three ingredients on Alice's sandwich is not found on Bob's sandwich). What's the probability that these sandwiches have exactly two ingredients in common?

4. **[25 points]** The $t^{th}$ day is a good day with probability $P(G_t) = p$; otherwise it is a bad day. Events $G_t$ and $G_s$ are independent for $t \neq s$.

   (a) **[6 points]** Let $X$ be the number of good days in any particular week. Suppose that $p = 0.5$. What is $P\{X \geq 2\}$?

   (b) **[6 points]** Let $D$ be the number of bad days that occur in a row, prior to the first good day (notice that $D = 0$ is a possibility, therefore $D$ is not a geometric random variable). Suppose that $p = 0.5$. What is $E[D^2]$?

   (c) **[6 points]** Let $Y$ be the number of good days that occur in some particular 100-day period, and assume that $p = 0.5$. Use the Chebyshev bound to find a lower bound on the quantity $P\{41 < Y < 59\}$.

(d) [**7 points**] Let $Z$ be the number of good days that occur in some particular $n$-day period. For this part of the problem, assume that $p$ is an unknown small number, and $n$ is an unknown large number, but assume that $E[Z] = 6$. Use the Poisson approximation to find $P\{Z \leq 2\}$.

5. **[25 points]** Let $X$ be the number of microphones carried by a spy. The number of microphones carried by a spy from Organization A is a random variable with the following pmf:

$$p_A(k) = \begin{cases} \frac{1}{55}(10-k) & 0 \le k \le 9 \\ 0 & \text{otherwise} \end{cases}$$

The number of microphones carried by a spy from Organization B is a random variable with the following pmf:

$$p_B(k) = \begin{cases} \frac{k}{55} & 1 \le k \le 10 \\ 0 & \text{otherwise} \end{cases}$$

Any given spy is from Organization A with probability $\pi_A = 0.8$, otherwise he or she is from Organization B.

(a) **[6 points]** You meet a spy. What is the probability that he or she is carrying no microphones?

(b) **[7 points]** You discover that the spy is carrying at least one microphone ($X \ge 1$). Given only this one piece of information, what is the probability that this spy is from Organization B?

(c) Your mission is to find spies, to count the number of microphones each spy is carrying, and to determine, with no other information available to you, to which organization each spy belongs (Organization A or Organization B). Specify the ML and MAP decision rules for this problem; break ties in favor of Organization B. In both cases, you should be able to specify a rule of the form "Choose Organization B if $X \geq d$, otherwise choose Organization A," for some particular value of $d$.

i. [**6 points**] ML Decision Rule:

ii. [**6 points**] MAP Decision Rule: