University of Illinois Spring 2012

ECE 313: Final Exam

Monday, May 7, 2012, 1:30 p.m. — 4:30 p.m.
150 Animal Sciences Lab (Sections E and C) & 116 Roger Adam Lab (Sections D and F)

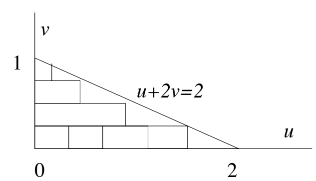
1. (a) The joinf pdf is the product of the pdfs of an exponential with parameter $\lambda = 2$ and an Erlang with parameters $r = 2$ and $\lambda = 1$. Therefore, $f_X(u) = ue^{-u}$ for $u \geq 0$ and zero else.
   Alternatively, $f_X(u) = \int_0^\infty 2ue^{-u-2v}dv = ue^{-u}$ for $u \geq 0$.

   (b) X and Y are independent. Therefore, for $u > 0$,

$$f_{Y|X}(v|u) = \begin{cases} 2e^{-2v} & \text{for } v \geq 0 \\ 0 & \text{for } v < 0 \end{cases}$$

   It's undefined for $u \leq 0$.
   An alternative approach is to use that for $u > 0$, and $v \geq 0$, $f_{Y|X}(v|u) = \frac{f_{X,Y}(u,v)}{f_X(u)} = \frac{2ue^{-u-2v}}{ue^{-u}} = 2e^{-2v}$.

   (c) The probability in question is the integral of the joint pdf over the region $\{u \geq 0, v \geq 0, u + 2v \leq 2\}$



   So $P\{X+2Y \leq 2\}$ can be expressed as $\int_0^2 \int_0^{1-u/2} 2ue^{-u-2v}dvdu$ or $\int_0^1 \int_0^{1-2v} 2ue^{-u-2v}dudv$ (Note: This can be shown to equal $1 - 5e^{-2}$.)

   (d) $\hat{E}[Y|X] = \mu_Y + \frac{\text{Cov}(X,Y)}{\text{Var}(X)}(X - \mu_X)$, and in this case, $\text{Cov}(X,Y) = 0$ because X and Y are indepedent. Therefore $\hat{E}[Y|X = 2] = \mu_Y = 1/2$.

2. (a) By the problem description we take X to have the binomial distribution with parameters $n = 10$ and $p = 0.5$. Since $S = 3X - 3(10 - X) = 6X - 30$,

$$P\{S \geq 12\} = P\{6X - 30 \geq 12\} = P\{X \geq 7\}.$$

   Since $E[X] = 5$, the Markov inequality yields:

$$P\{S \geq 12\} = P\{X \geq 7\} \leq \frac{E[X]}{7} = \frac{5}{7}.$$

   (b) By the observed symmetry and the connection to X,

$$P\{S \geq 12\} = (0.5)P\{|S| \geq 12\} = (0.5)P\{|X - 5| \geq 7\}.$$

   Since $E[X] = 5$ and $\text{Var}(X) = 10(0.5)(0.5) = \frac{5}{2}$, Chebyshev inequality yields

$$P\{S \geq 12)\} = (0.5)P\{|X - 5| \geq 2\} \leq \frac{\text{Var}(X)}{2 \cdot 4} = \frac{5}{16}.$$

(c)

$$P\{S \geq 12\} = P\{X \geq 7\} = P\{X \geq 6.5\} = P\left\{\frac{X-5}{\sqrt{2.5}} \geq \frac{1.5}{\sqrt{2.5}}\right\} \approx Q\left(\frac{1.5}{\sqrt{2.5}}\right).$$

3. (a) The likelihood ratio is given by $\Lambda(k) = \frac{p_1(k)}{p_0(k)} = \frac{\binom{n}{k}(\frac{2}{3})^k(\frac{1}{3})^{72-k}}{\binom{n}{k}(\frac{1}{3})^k(\frac{2}{3})^{72-k}} = 2^{2k-72}$. The ML rule declares H_1 to be the hypothesis if $\Lambda(k) \geq 1$, or simply, if $k \geq 36$.

(b) $p_{false\ alarm} = P(X \geq 34|H_0 \text{ true})$. By the De Moivre-Laplace special case of the central limit theorem, given that H_0 is true, we know that X is approximately Gaussian with mean $np = \frac{72}{3} = 24$ and variance $\sigma^2 = np(1-p) = 72\frac{1}{3}\frac{2}{3} = 16$. Therefore

$$p_{false\ alarm} = P(X \geq 34|H_0 \text{ true}) = P\left(\frac{X-24}{4} \geq \frac{34-24}{4}\middle| H_0 \text{ true}\right) \approx Q\left(\frac{10}{4}\right) = Q(2.5).$$

(c) Since $\frac{\pi_0}{\pi_1} = 9$, the MAP rule declares H_1 to be the hypothesis if $\Lambda(k) \geq 9$, or equivalently, if $2k - 72 \geq 4$ or $k \geq 38$.

(d)

$$\begin{aligned} P(H_0|X=38) &= \frac{P\{H_0, X=38\}}{P\{X=38\}} = \frac{(0.9)p_0(38)}{(0.9)p_0(38) + (0.1)p_1(38)} \\ &= \frac{0.9}{0.9 + (0.1)\Lambda(38)} = \frac{0.9}{0.9 + (0.1)16} = \frac{0.9}{2.5} = 0.36. \end{aligned}$$

4. (a) $E[X^2] = \int_0^3 \frac{u^2}{3} du = \left.\frac{u^3}{9}\right|_0^3 = 3.$

(b) $P\{\lfloor X^2 \rfloor = 3\} = P\{3 \leq X^2 < 4\} = P\{\sqrt{3} \leq X < 2\} = \frac{2-\sqrt{3}}{3}$.

(c) The range of Y is $(-\infty, \ln 3]$. Thus,

$$F_Y(c) = \begin{cases} P\{\ln X \leq c\} = P\{X \leq e^c\} = \frac{e^c}{3} & -\infty < c \leq \ln 3 \\ 1 & c \geq \ln 3 \end{cases}.$$

5. Since $\text{Var}(X_i) = \text{Var}(Y_j) = 4$ for all i, j, the given values of the correlation coefficients translate to

$$\text{Cov}(X_i, Y_j) = \begin{cases} 3 & \text{if } i = j \\ -1 & \text{if } |i-j| = 1 \\ 0 & \text{else.} \end{cases}$$

Therefore,

$$\begin{aligned} \text{Cov}(W, Z) &= \text{Cov}(\sum_{i=1}^n X_i, \sum_{j=1}^n Y_i) \\ &= \sum_{i=1}^n \text{Cov}(X_i, Y_i) + \sum_{i=1}^{n-1} \text{Cov}(X_i, Y_{i+1}) + \sum_{i=2}^n \text{Cov}(X_i, Y_{i-1}) \\ &= 3n - 2(n-1) = n + 2 \end{aligned}$$

6. (a) Since X and $X+Y$ are independent, they are uncorrelated. So $0 = \text{Cov}(X, X+Y) = \text{Cov}(X,X) + \text{Cov}(X,Y) = \text{Var}(X) + \text{Cov}(X,Y) = 1 + \text{Cov}(X,Y)$. Therefore, $\text{Cov}(X,Y) = -1$.
   (b) Since X and $X+Y$ are independent, $E[X|X+Y=2] = E[X] = 0$.
   (c) Since X and Y are jointly Gaussian, $E[Y|X=2] = \widehat{E}[Y|X=2] = \mu_Y + \frac{\text{Cov}(Y,X)}{\text{Var}(X)}(2-\mu_X) = -2$.

7. (a) Since there are four cards of each rank, all ranks are equally likely to appear on the card drawn. So

$$p_X(i) = \begin{cases} 4/13 & i=0 \\ 5/13 & i=1 \\ 3/13 & i=2 \\ 1/13 & i=3 \\ 0 & else. \end{cases}$$

   (b) $E[X] = \sum_{i=0}^{3} i p_X(i) = (0)\frac{4}{13} + (1)\frac{5}{13} + (2)\frac{3}{13} + (3)\frac{1}{13} = \frac{14}{13}$
   (c) Using LOTUS, we have

$$E[\sin(X\pi/2)] = \sum_{i=0}^{3} \sin(i\pi/2) p_X(i) = \sin(0)\frac{4}{13} + \sin(\pi/2)\frac{5}{13} + \sin(2\pi/2)\frac{3}{13} + \sin(3\pi/2)\frac{1}{13} = \frac{4}{13}$$

8. (a) $C \in \{0, 10, 20\}$.
   (b) Let F_i be the event that link i fails, then $P(F_i) = p_i$.
   $p_X(0) = P(F_3(F_1 \cup F_2) \cup F_4) = p_3(p_1+p_2-p_1p_2) + p_4 - p_3(p_1+p_2-p_1p_2)p_4$.
   Note that $p_X(10) = P(((F_1^cF_2^c)\triangle F_3^c)F_4)$ where $A\triangle B = (AB^c)\cup(A^cB)$ is the symmetric difference between sets A and B, and $P(A\triangle B) = P(A) + P(B) - 2P(AB)$. Thus,
   $p_X(10) = ((1-p_1)(1-p_2) + (1-p_3) - 2(1-p_1)(1-p_2)(1-p_3))(1-p_4)$.
   $p_X(20) = P(F_1^cF_2^cF_3^cF_4^c) = (1-p_1)(1-p_2)(1-p_3)(1-p_4)$.
   Of course if expressions for any two of the above three are found, we could use the fact the probabilities sum to one to find the third one.
   (c) $p_X(0) = 11/16, p_X(20) = 1/16, p_X(10) = 4/16$.

9. (a) The event $\{T_2 \geq 30\}$ is the same as the event that there are less than two arrivals, i.e. either zero or one arrivals, in the first thirty minutes. The number of arrivals in the first thirty minutes has the Poisson distribution with mean $30\lambda = 3$. Therefore, $P\{T_2 \geq 30\} = e^{-3} + 3e^{-3} = 4e^{-3}$.
   (b) T_2 is an Erlang random variable with parameters $r = 2$, $\lambda = 0.1$; its expected value is $E[T_2] = r/\lambda = 20$ minutes.
   (c) A Poisson process over nonoverlapping periods of time is independent, therefore the random variable $T_2 - U_1$ is exponential, and

$$f_{T_2|U_1}(t|u) = \begin{cases} (0.1)e^{-0.1(t-u)} & t > u \\ 0 & \text{else} \end{cases}$$

   (d)

$$f_{U_1|T_2}(u|t) = \frac{f_{U_1,T_2}(u,t)}{f_{T_2}(t)} = \begin{cases} \frac{(0.1)e^{-(0.1)u}(0.1)e^{-(0.1)(t-u)}}{(0.1)^2te^{-(0.1)t}} = \frac{1}{t} & 0 < u < t \\ 0 & \text{otherwise} \end{cases}.$$

   That is, given the second fish jumped at time t, the time the first fish jumped is uniformly distributed over $[0,t]$.

10. (a) $P\{Y \le -\frac{1}{3}\} = \int_{-1}^{-\frac{1}{3}} \int_{-1}^{1} \frac{uv+1}{4} du dv = \int_{-1}^{-\frac{1}{3}} \frac{1}{2} dv = \frac{1}{3}$.

(b) We first find the marginal pdf of Y: if $-1 \le v \le 1$,

$$f_Y(v) = \int_{-1}^{1} \frac{1}{4}(uv+1)du = \frac{1}{2}.$$

Otherwise, $f_Y(v) = 0$. Thus, Y is a uniform random variable over the interval $[-1, 1]$. Therefore, Y has mean zero and $E[Y^2] = \mathrm{Var}(Y) = \frac{2^2}{12} = \frac{1}{3}$. Thus, $\delta^* = E[Y] = 0$, and $MSE = \mathrm{Var}(Y) = \frac{1}{3}$.

(c) By symmetry, X and Y have the same distribution. So X is uniformly distributed on the interval $[-1, 1]$ and $g(u_0)$ must be identified for $u_0 \in [-1, 1]$. For such u_0, the conditional pdf of Y is

$$f_{Y|X}(v|u_0) = \begin{cases} \frac{f_{X,Y}(u_0,v)}{f_X(u_0)} = \frac{1}{2}(u_0 v + 1) & -1 \le v \le 1 \\ 0 & |v| > 1 \end{cases}$$

Thus,

$$g^*(u_0) = E[Y|X = u_0] = \int_{-1}^{1} v \cdot \frac{1}{2}(u_0 v + 1)dv = \frac{u_0}{2} \int_{-1}^{1} v^2 dv = \frac{u_0}{3}.$$

Thus, $g^*(X) = E[Y|X] = \frac{X}{3}$. Furthermore,

$$\begin{aligned} MSE(g^*) &= E[(Y - g^*(X))^2] = E[Y^2] - E[(g^*(X))^2] = E[Y^2] - E\left[\left(\frac{X}{3}\right)^2\right] \\ &= \frac{1}{3} - \frac{1}{9}E[X^2] = \frac{1}{3} - \frac{1}{27} = \frac{8}{27}. \end{aligned}$$

(d) Since the unconstrained estimator g^* is given by a linear function, $L^*(X) = g^*(X) = \frac{X}{3}$, and the corresponding MSE is the same: $\frac{8}{27}$.
Another Solution: Since X and Y have mean zero,

$$\mathrm{Cov}(X, Y) = E[XY] = \int_{-1}^{1} \int_{-1}^{1} uv \cdot \frac{1}{4}(uv + 1)du dv = \frac{1}{4} \int_{-1}^{1} \int_{-1}^{1} (u^2 v^2 + uv)du dv = \frac{1}{9}.$$

Thus,

$$L^*(X) = E[Y] + \frac{\mathrm{Cov}(X, Y)}{\mathrm{Var}(X)}(X - E[X]) = \frac{X}{3}$$

and

$$MSE(L^*) = \mathrm{Var}(Y) - \frac{\mathrm{Cov}(X, Y)^2}{\mathrm{Var}(X)} = \frac{1}{3} - \frac{\left(\frac{1}{9}\right)^2}{\frac{1}{3}} = \frac{1}{3} - \frac{1}{27} = \frac{8}{27}.$$

11. (a) False, True, False
(b) False, True,
(c) False, True, True
(d) True, True