University of Illinois Spring 2012

ECE 313: Final Exam

Monday, May 7, 2012, 1:30 p.m. — 4:30 p.m.
150 Animal Sciences Lab (Sections E and C) & 116 Roger Adam Lab (Sections D and F)

1. [**25 points**] Let X and Y be two random variables with joint pdf

$$f_{X,Y}(u,v) = \begin{cases} 2ue^{-u-2v} & u \geq 0, v \geq 0, \\ 0 & else. \end{cases}$$

   (a) [**7 points**] Find the marginal pdf of X.
   (b) [**6 points**] Find the conditional pdf $f_{Y|X}(v|u)$. Be sure to include specifying what values of u, v it is well defined for and what values it is zero for.
   (c) [**6 points**] Carefully write down an expression for $P\{X + 2Y \leq 2\}$ involving integration over a region in the plane. Set it up as an iterated integral, clearly showing the limits of integration. *You do not need to calculate the integral.*
   (d) [**6 points**] Find $\hat{E}[Y|X = 2]$.

2. [**20 points**] At the end of this exam, there are ten True/False questions. You will get 3 points for each correct answer, -3 points for each wrong answer, and zero points for any unanswered question. Assume that you have no idea what the correct answers are, but attempt to answer all ten questions by randomly guessing True or False on each one. Let S be your total score from these T/F questions.

   (a) [**6 points**] Find the upper bound on $P\{S \geq 12\}$ yielded by the Markov inequality. Since the Markov inequality is for nonnegative random variables, apply it to the random variable X, where X is the number of correct answers.
   (b) [**6 points**] Find the upper bound on $P\{S \geq 12\}$ yielded by the Chebychev inequality and the observation that, by symmetry, $P\{S \geq 12\} = P\{S \leq -12\}$. (Use this observation to make the bound tighter.)
   (c) [**8 points**] Express the approximate value of $P\{S \geq 12\}$ in terms of the Q function based on the Gaussian approximation suggested by the central limit theorem. To be definite, *use the continuity correction.*

3. [**20 points**] Consider a binary hypothesis testing problem with observation X. Under H_0, X has the binomial distribution with parameters $n = 72$ and $p = \frac{1}{3}$. Under H_1, X has the binomial distribution with parameters $n = 72$ and $p = \frac{2}{3}$.

   (a) [**5 points**] Describe the maximum likelihood decision rule for an observation k, where k is an arbitrary integer with $0 \leq k \leq 72$. Express the rule in terms of k as simply as possible. To be definite, in case of a tie in likelihoods, declare H_1 to be the hypothesis.
   (b) [**5 points**] Suppose a particular decision rule declares that H_1 is the true hypothesis if and only if $X \geq 34$. Express the approximate value of $p_{false\ alarm}$ for this rule in terms of the Q function, where $Q(c) = \int_c^\infty \frac{1}{\sqrt{2\pi}} \exp(-u^2/2) du$. (To be definite, don't use the continuity correction.)
   (c) [**5 points**] Describe the MAP decision rule for an observation k, where k is an arbitrary integer with $0 \leq k \leq 72$, for the prior distribution $\pi_0 = 0.9$ and $\pi_1 = 0.1$. Express the rule in terms of k as simply as possible.
   (d) [**5 points**] Assuming the same prior distribution as in part (c), find $P(H_0|X = 38)$.

4. **[20 points]** Suppose X is uniformly distributed on the interval [0,3].

   (a) **[5 points]** Find $E[X^2]$.

   (b) **[7 points]** Find $P\{\lfloor X^2 \rfloor = 3\}$, where $\lfloor v \rfloor$ is the greatest integer less than or equal to v.

   (c) **[8 points]** Find the cumulative distribution function (CDF) of $Y = \ln X$. Be sure to specify it over the entire real line.

5. **[10 points]** Suppose $X_1, \ldots X_n$ and $Y_1, \ldots, Y_n$ are random variables on a common probability space such that $\mathrm{Var}(X_i) = \mathrm{Var}(Y_i) = 4$ for all i, and

$$\rho_{X_i,Y_j} = \begin{cases} 3/4 & \text{if } i = j \\ -1/4 & \text{if } |i-j| = 1 \\ 0 & \text{else.} \end{cases}$$

   Let $W = \sum_{i=1}^n X_i$ and $Z = \sum_{i=1}^n Y_i$. Express $\mathrm{Cov}(W, Z)$ as a function of n.

6. **[15 points]** Suppose X and Y have a bivariate Gaussian joint distribution with $E[X] = E[Y] = 0$ and $\mathrm{Var}(X) = 1$. (The variance of Y and the correlation coefficient are not given.) Finally, suppose X is independent of $X + Y$.

   (a) **[5 points]** Find $\mathrm{Cov}(X, Y)$.

   (b) **[5 points]** Find $E[X|X + Y = 2]$.

   (c) **[5 points]** Find $E[Y|X = 2]$.

7. **[15 points]** Consider a standard deck of 52 cards, and choose one card at random. Let X take the following values based on the rank of the card chosen:

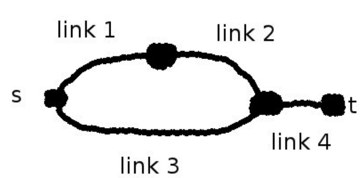
$$X = \begin{cases} 0 & \text{rank is } A, K, Q, J \\ 1 & \text{rank is } 10, 9, 8, 7, 6 \\ 2 & \text{rank is } 5, 4, 3 \\ 3 & \text{rank is } 2. \end{cases}$$

   (a) **[5 points]** Find the pmf of X.

   (b) **[5 points]** Find $E[X]$.

   (c) **[5 points]** Find $E[\sin(X\pi/2)]$.

8. **[20 points]** Consider the following $s - t$ flow network, where link $i \in \{1, 2, 3, 4\}$ fails with probability p_i. Let c_i be the flow capacity of link i, then $c_1 = 10$, $c_2 = 20$, $c_3 = 10$, $c_4 = 30$.



   (a) **[5 points]** What values can the capacity C of this network take?

   (b) **[10 points]** Find the distribution (pmf) of its capacity in terms of the p_i's. Do *not* rely on the fact that the sum over the pmf is one. That is, for each possible value u of the capacity, find an expression for $p_X(u)$ that is not simply one minus the sum of the other probabilities.

   (c) **[5 points]** Let $p_i = 1/2$ for all i. Obtain the numerical values for the pmf of the capacity (simplify as much as you can).

9. **[25 points]** Joe fishes with a net. Joe's strategy is unusual: he waits for two fish to jump, then throws the net immediately after seeing the second fish jump. Fish jump according to a Poisson process with parameter $\lambda = 0.1$ jumps per minute. Assume that Joe begins waiting at time zero; the first fish jumps at time U_1, and the second fish jumps at time T_2. In all parts of this problem, you may leave numerical powers of e in your answer.

   (a) **[7 points]** What's the probability that $T_2 \geq 30$ minutes?

   (b) **[6 points]** What is $E[T_2]$?

   (c) **[6 points]** Suppose that Joe sees the first fish jump at time $U_1 = u$ minutes. What is $f_{T_2|U_1}(t|u)$?

   (d) **[6 points]** Joe's friend doesn't see the first fish jump, but sees Joe throw the net immediately after the second fish jumps at $T_2 = t$ minutes. What is $f_{U_1|T_2}(u|t)$?

10. **[25 points]** Suppose two random variables X and Y have the following joint pdf:

$$f_{X,Y}(u,v) = \begin{cases} \dfrac{uv+1}{4}, & \text{if } -1 \leq u \leq 1 \text{ and } -1 \leq v \leq 1, \quad (1) \\ 0, & \text{elsewhere.} \quad (2) \end{cases}$$

   (a) **[5 points]** Find the numerical value of $P\{Y \leq -\frac{1}{3}\}$.

   (b) **[5 points]** Find the constant estimator δ^* of Y and the corresponding mean square error (MSE).

   (c) **[10 points]** Find the unconstrained estimator $g^*(X)$ for observation X and the corresponding MSE. (Hint: The MSE must be deterministic, not random.)

   (d) **[5 points]** Find the linear estimator $L^*(X)$ for observation X and the corresponding MSE.

11. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

   (a) Suppose X, Y, Z are independent, Bernoulli($\frac{1}{2}$) random variables.

TRUE	FALSE	
☐	☐	$X + Y$ is independent of $X - Y$.
☐	☐	$\mathrm{Cov}(X + 2Y, 2X - Y) = 0$.
☐	☐	$\mathrm{Cov}(XY, XZ) = \frac{1}{8}$

   (b) Let X and Y be two independent random variables, each with zero mean and finite variance, and let a, b be two real constants.

TRUE	FALSE	
☐	☐	$P\{\{X \leq a\} \cup \{Y \leq b\}\} = P\{X \leq a\} + P\{Y \leq b\}$
☐	☐	$E\left[(X+Y)^2\right] = \mathrm{Var}(X) + \mathrm{Var}(Y)$

   (c) Let X be a random variable with an even function $f_X(u)$ as its pdf, i.e. $f_X(u) = f_X(-u)$ for all u. Assume $E[|X|^3]$ is finite.

TRUE	FALSE	
☐	☐	$\text{Var}(X) = \text{Var}(\vert X\vert)$.
☐	☐	$E[(1-X)^2] = 1 + \text{Var}(X)$
☐	☐	$E[(1-X)^3] = 1 + 3\text{Var}(X)$.

(d) For two random variables X and Y, let MSE_C, MSE_U, and MSE_L be the mean square errors for the constant estimator δ^*, the unconstrained estimator $g^*(X)$, and the linear estimator $L^*(X)$, respectively, of Y for observation X.

TRUE	FALSE	
☐	☐	$MSE_C \geq MSE_L \geq MSE_U$.
☐	☐	If X and Y are independent, $MSE_C = MSE_L = MSE_U$.