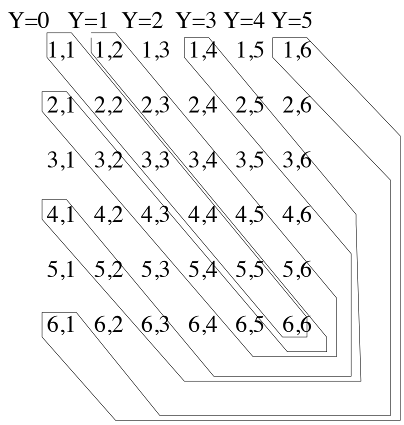
University of Illinois Fall 2012

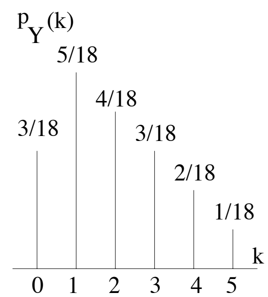
ECE 313: Hour Exam I

Monday February 27, 2012
7:00 p.m. — 8:15 p.m.
163 Everitt, 269 Everitt, 103 Talbot, & 165 Everitt

1. (a) Clear Y is nonnegative integer valued. The largest possible value is 5, which occurs when $X_1 = 1$ and $X_2 = 6$ (or vice versa). So the support (i.e. set of possible values) of Y is $\{0, 1, 2, 3, 4, 5\}$.

   (b) Of the 36 equally likely possible values of (X_1, X_2), it is easy to count how many give rise to each possible value of Y. The number of ways to have Y=0,1,2,3,4, or 5 are 6,10,8,6,4, or 2, respectively. This gives p_Y as shown.



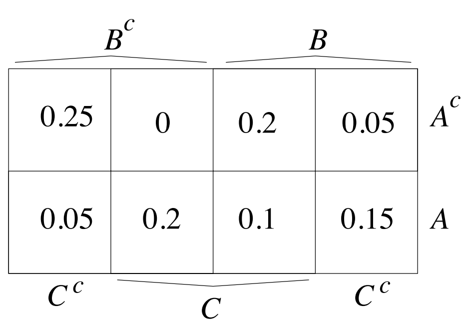


   (c) $E[Y] = \sum_{k=0}^{5} k \cdot p_Y(k) = \frac{1\cdot5+2\cdot4+3\cdot3+4\cdot2+5\cdot1}{18} = \frac{35}{18}$

   (d) (You may have two terms in your answer–you don't need to simplify to the end.) By LOTUS, $E[Y^2] = \sum_{k=0}^{5} k^2 \cdot p_Y(k) = \frac{1^2\cdot5+2^2\cdot4+3^2\cdot3+4^2\cdot2+5^2\cdot1}{18} = \frac{105}{18} = \frac{35}{6}$. Therefore, $\mathrm{Var}(Y) = E[Y^2] - E[Y]^2 = \frac{35}{6} - \left(\frac{35}{18}\right)^2 \quad (\approx 6 - 2^2 = 2)$

   (e) By LOTUS, or linearity of expectation, $E[Z] = 100E[Y]$. Since variance scales as the square of a multiplicative factor, $\mathrm{Var}(Z) = 10^4\mathrm{Var}(Y)$.

2. The independence of A and B and the fact $P(A) = P(B) = 0.5$ implies that $P(AB) = P(AB^c) = P(A^cB) = P(A^cB^c) = 0.25$. Then, from $P(ABC)$, one can find $P(ABC^c)$. From $P(BC)$, one can find $P(A^cBC)$ and $P(A^cBC^c)$. From $P(AC)$ one can find $P(AB^cC)$ and $P(AB^cC^c)$. Finally, from $P(C)$ one can find the rest.



3. (a) There are $\binom{7}{3} = \frac{7\cdot6\cdot5}{3\cdot2\cdot1} = 35$ ways to select the four people, and the number of ways to select a team with Alice on it is the number of ways to select three of the other six people

to go with Alice, or $\binom{6}{3} = \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1} = 20$. So $P(A) = \frac{20}{35} = \frac{4}{7}$. ALTERNATIVELY, a faction 4/7 of the entire debate team is selected so by symmetry, each person has a probability 4/7 of being selected.

(b) Given Bob is selected, there are 20 ways to fill out the rest of the team, but if Alice is also selected, there are $\binom{5}{2} = 10$ ways to fill out the rest of the team, so $P(A|B) = \frac{10}{20} = 0.5$. ALTERNATIVELY by the definition, $P(A|B) = \frac{P(AB)}{P(B)} =$ (number of selections including both Alice and Bob)/(number of selections including Bob) $= \frac{10}{20} = 0.5$. OR ALTERNATIVELY given Bob is selected, half of the other six people are also selected so by symmetry, Alice has chance 0.5 to be among them.

(c) $P(A \cup B) = P(A) + P(B) - P(AB) = \frac{20}{35} + \frac{20}{35} - \frac{10}{35} = \frac{6|}{7}$. ALTERNATIVELY, the number of selections that exclude both Alice and Bob is the number of ways to select four out of the other five people, which is 5. So $P(A \cup B) = 1 - P((A \cup B)^c) = 1 - \frac{5}{35} = \frac{6}{7.}$

4. (a) The distribution of X is approximately Poisson with parameter $\lambda = np = 3$. So $P\{X \geq 2\} = 1 - p_X(0) - p_X(1) \approx 1 - \frac{e^{-\lambda}\lambda^0}{0!} - \frac{e^{-\lambda}\lambda^1}{1!} = 1 - e^{-3} - 3e^{-3} = 1 - 4e^{-3}$.

(b) The half-width of the confidence interval is $0.025 = \frac{a}{2\sqrt{n}} = \frac{a}{2\sqrt{10{,}000}}$, so $a = 5$. Therefore, we can claim a confidence level of $1 - \frac{1}{a^2} = 1 - \frac{1}{5^2} = 0.96$.

(c) The likelihood (i.e. probability) of observing $X = 7$ is zero if $n \leq 6$. If $n \geq 7$, the likelihood is given by $p_X(7) = \binom{n}{7}(0.03)^7(0.97)^{n-7}$. The desired estimate $\hat{n}_{ML}$ is the value of n that maximizes this. That is, $\hat{n}_{ML}$ is the value of n that maximizes $L(n) = \binom{n}{7}(0.03)^7(0.97)^{n-7}$ over the range $n \geq 7$. Consider the ratio:

$$\frac{L(n)}{L(n-1)} = \frac{\binom{n}{7}(0.97)}{\binom{n-1}{7}} = \frac{n(0.97)}{n-7}$$

Note that $\frac{L(n)}{L(n-1)} > 1$ if $n(0.97) > n - 7$, or $7 \geq (0.03)n$, or $n < 233.33$. Similarly, $\frac{L(n)}{L(n-1)} < 1$ if $n > 233.33$. So $L(n)$ is strictly increasing in the range $7 \leq n \leq 233$ and strictly decreasing in the range $233 \leq n < \infty$. So $\hat{n}_{ML} = 233$. (For general p and observed value k of X, an ML estimate of n is given by $\left\lfloor \frac{k}{p} \right\rfloor$.)

5. (a) By the formula for the geometric distribution, for $k \geq 1$, the probability $X = k$ if Aristotle is on duty is given by $p_A(k) = \frac{2}{3}\left(\frac{1}{3}\right)^{k-1}$, and the probability $X = k$ if Bellatrix is on duty is given by $p_B(k) = \frac{1}{2}\left(\frac{1}{2}\right)^{k-1} = 2^{-k}$.

(i) The probability of error given Aristotle is on duty is $1 - p_A(1) = 1 - \frac{2}{3} = \frac{1}{3}$.

(ii) The probability of error given Bellatrix is on duty is $p_B(1) = \frac{1}{2}$.

(iii) Let $\pi_A = 1 - \pi_B = 1/3$. The overall probability of error is π_A times the answer to (i) plus π_B times the answer to (ii), or $p_e = (1/3)(1/3) + (2/3)(1/2) = 4/9$.

(b) The maximum likelihood decision rule is to say Bellatrix if X takes on a value k such that:

$$1 < \frac{p_B(k)}{p_A(k)} = \frac{(1/2)^k}{(2/3)(1/3)^{k-1}} = \frac{1}{2}\left(\frac{3}{2}\right)^k$$

The likelihood ratio $\frac{1}{2}\left(\frac{3}{2}\right)^k$ is strictly increasing in k and its values for $k = 1, 2$ are $\frac{3}{4}$, $\frac{9}{8}$, respectively, So $k_{ML} = 2$. That is, the rule in part (a) is the ML rule.