University of Illinois Fall 2012

ECE 313: Hour Exam I

Monday February 27, 2012
7:00 p.m. — 8:15 p.m.
163 Everitt, 269 Everitt, 103 Talbot, & 165 Everitt

1. [**25 points**] Suppose two students each roll a standard, six-sided die. Let X_1 be the number showing for the first student and X_2 be the number showing for the second student. Let Y be the absolute value of the difference: $Y = |X_1 - X_2|$.

   (a) [**2 points**] Identify the set of possible values of Y.

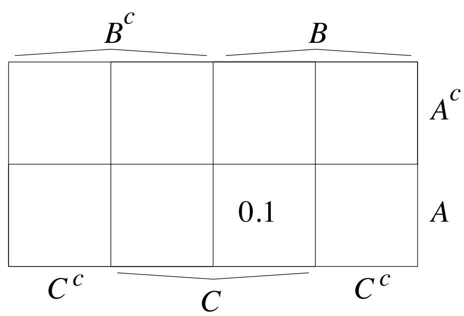
   (b) [**8 points**] Find and carefully sketch the pmf of Y.

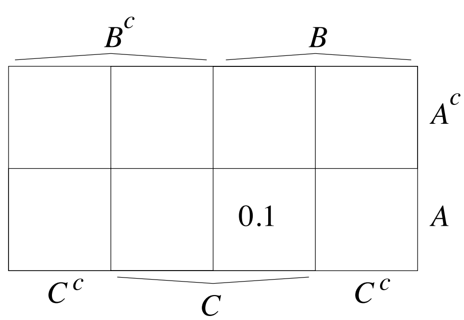
   (c) [**5 points**] Find $E[Y]$.

   (d) [**5 points**] Find $\mathrm{Var}(Y)$. (You may have two terms in your answer–you don't need to simplify to the end.)

   (e) [**5 points**] Suppose $Z = 100Y$, where Y is defined above. Express the mean and variance of Z in terms of the mean and variance of Y. (Note: This does NOT require you to solve the previous parts of this problem.)

2. [**10 points**] Suppose A, B, and C are events for a probability experiment such that A and B are mutually independent, $P(A) = P(B) = P(C) = 0.5$, $P(AC) = P(BC) = 0.3$, and $P(ABC) = 0.1$. Fill in the probabilities of all events indicated in the Karnaugh map provided. Show your work. An extra copy is provided, so circle the one with your final answer.





3. [**15 points**] Suppose Alice and Bob are among seven people on a debate team. Suppose four of the team members are selected to travel to a debate, with all sets of four having equal probability. Let A be the event that Alice is among the four selected, and B be the event Bob is among the four selected. Find the probabilities indicated. *Briefly explain your reasoning.*

   (a) [**5 points**] $P(A)$.

(b) **[5 points]** $P(A|B)$.

(c) **[5 points]** $P(A \cup B)$.

4. **[25 points]** Suppose X has the binomial distribution with parameters n and p.

   (a) **[8 points]** Suppose (for this part only) $p = 0.03$ and $n = 100$. Find the Poisson approximation to $P\{X \geq 2\}$. (You may leave one or more powers of e in your answer, but not an infinite number of terms.)

   (b) **[8 points]** Suppose (for this part only) p is unknown and $n = 10,000$, and based on observation of X we want to estimate p within 0.025. That is, we will use a confidence interval with half-width 0.025. What's the largest confidence level we can claim? (The confidence level is the probability, from the viewpoint before X is observed, that the confidence interval will contain the true value p.)

   (c) **[9 points]** Suppose (for this part only) it is known that $p = 0.03$, but n is unknown. The parameter n is to be estimated. Suppose it is observed that $X = 7$. Find the maximum likelihood estimate $\hat{n}_{ML}$. (Hint: It is difficult to differentiate with respect to the integer parameter n, so another approach is needed to identify the minimizing value. Think about how a function on the integers behaves near its maximum. How can you tell whether the function is increasing or decreasing from one integer value to the next?)

5. **[25 points]** Everitt Lab employs two guards, named Aristotle and Bellatrix. Bellatrix guards the building on any given night with probability $\pi_B = 2/3$; Aristotle guards the building on other nights. Your job is to provide donuts. Let X be the number of donuts consumed per hour. If Aristotle is on duty, X is a geometric random variable with parameter $p = 2/3$. If Bellatrix is on duty, X is a geometric random variable with parameter $p = 1/2$. You have no way to find out who is on duty except by observing the number of donuts consumed.

   (a) **[15 points]** Suppose you use the following rule to decide who is on duty tonight:

   - If $X \geq 2$, say "Bellatrix."
   - If $X = 1$, say "Aristotle."

   i. What is the probability error for this rule, given Aristotle is working?

   ii. What is the probability of error for this rule, given Bellatrix is working?

   iii. What is the overall probability of error for this rule?

   (b) **[10 points]** The maximum likelihood decision rule for this hypothesis testing problem can be stated as:

   - If $X \geq k_{ML}$, say "Bellatrix."
   - Otherwise, say "Aristotle."

   for some positive integer value k_{ML}. Find the value of k_{ML}.