University of Illinois **Spring 2011**

ECE 313: Problem Set 12
Binary hypothesis testing for continuous-type random variables, joint cdfs, joint pmfs, and joint pdfs

Due: Wednesday April 20 by 4 p.m.
Reading: *313 Course Notes* Sections 3.10–4.3

1. [**Binary hypothesis testing for continuous-type random variables**]
Suppose that a transmitter can use two different signals to communicate with the receiver. Under hypothesis 0, the transmitter chooses to use the signal of type 0, which at the receiver's end is a random variable X with a Normal distribution with mean 1 and variance 1. On the other hand, under hypothesis 1, the transmitter chooses to use the signal of type 1, which at the receiver's end is a random variable X with a Normal distribution with mean 2 and variance 2.

   (a) State the ML decision rule in terms of a threshold test on the observed value of X^2 instead of a test that involves comparing the likelihood ratio to a threshold.

   (b) State the MAP decision rule in terms of a threshold test on the observed value of X^2 instead of a test that involves comparing the likelihood ratio to a threshold.

   (c) For what range (if any) of values of π_0, does the MAP decision rule always chooses hypothesis 0 (no matter what the observed value of the random variable is)?

   (d) Calculate the false alarm, missed detection and average error probabilities for the ML decision rule assuming $\pi_0 = 1/3$.

   (e) Calculate the false alarm, missed detection and average error probabilities for the MAP decision rule assuming $\pi_0 = 1/3$.

2. [**Binary hypothesis testing for continuous-type random variables**]
A manufacturer of resistors has two factories and the resistors are guaranteed to have a resistance value within 2Ω of the nominal value. Under hypothesis 0, resistors are manufactured at factory 0 and have a variation around the nominal value that is a random variable X which is uniformly distributed between $(-2, 2)$, whereas under hypothesis 1, resistors are manufactured at factory 1 and have a variation around the nominal value that is a random variable X with pdf given by $f_X(u) = \frac{1}{4}(2 - |u|)$ for $u \in (-2, 2)$ and zero else.

   (a) State the ML decision rule in terms of a threshold test on the observed value of $|X|$ instead of a test that involves comparing the likelihood ratio to a threshold.

   (b) State the MAP decision rule in terms of a threshold test on the observed value of $|X|$ instead of a test that involves comparing the likelihood ratio to a threshold.

   (c) For what range (if any) of values of π_0, does the MAP decision rule always chooses hypothesis 0 (no matter what the observed value of the random variable is)?

   (d) Calculate the false alarm, missed detection and average error probabilities for the ML decision rule assuming $\pi_0 = 1/3$.

   (e) Calculate the false alarm, missed detection and average error probabilities for the MAP decision rule assuming $\pi_0 = 1/3$.

3. [**Joint cumulative distribution functions**]
Consider the following function

$$F(u, v) = \begin{cases} 0 & u + v \leq 1 \\ 1 & u + v > 1. \end{cases}$$

Is this a valid joint CDF. Why or why not? Prove your answer and show your work.

4. **[Joint probability mass functions]**
Suppose that two cards are drawn at random from a deck of 52 cards. Let X be the number of queens obtained and let Y be the number of spades obtained.

   (a) Find the joint probability mass function of X and Y.
   (b) Find the marginal probability mass function of X.
   (c) Find the marginal probability mass function of Y.
   (d) Find $P(X = Y)$
   (e) Find $P(X \leq Y)$
   (f) Find $P(X = 2|Y = 2)$.

5. **[Joint probability mass functions]**
Suppose that you have a fair die and a fair coin. You first roll your die, and let X be the number showing. Then you flip your coin X times and let Y be the number of heads showing.

   (a) Find the joint probability mass function of X and Y.
   (b) Find the marginal probability mass function of X.
   (c) Find the marginal probability mass function of Y.
   (d) Find $P(X \neq Y)$
   (e) Find $P(Y = 3|X = 6)$.

6. **[Joint probability density functions]**
The jointly continuous random variables X and Y have joint pdf

$$f_{X,Y}(u, v) = \begin{cases} 1.5 & 0 \leq u < 1, 0 \leq v < 1, 0 \leq u + v < 1, \\ 0.5, & 0 \leq u < 1, 0 \leq v < 1, 1 \leq u + v < 2, \end{cases}$$

and zero elsewhere.

   (a) Find the marginal pdf of Y.
   (b) Find $P(X + Y \geq 3/2)$.
   (c) Find $P(X^2 + Y^2 \leq 1)$.