University of Illinois Spring 2011

ECE 313: Problem Set 6

Due: Wednesday, March 2nd at 4 p.m.
Reading: Class Notes, Hypothesis Testing.

This Problem Set contains six problems.

1. **[Decision Making]**
If H_0 is the true hypothesis, the random variable X takes on values 0, 1, 2, and 3 with probabilities 0.1, 0.2, 0.3, and 0.4 respectively. If H_1 is the true hypothesis, the random variable X takes on values 0, 1, 2, and 3 with probabilities 0.4, 0.3, 0.2, and 0.1 respectively.

   (a) Write down the likelihood matrix L and indicate the *maximum-likelihood decision rule* by shading the appropriate entries in L. What is the false-alarm probability P_{FA} and what is the missed-detection probability P_{MD} for the maximum-likelihood decision rule?

   (b) Suppose that the hypotheses have *a priori* probabilities $\pi_0 = 0.7$ and $\pi_1 = 0.3$. Use the law of total probability to find the average error probability of the maximum-likelihood decision rule that you found in part (a).

   (c) Use the *a priori* probabilities given in part (b) to find the joint probability matrix J and indicate on it the Bayesian decision rule, which is also known as the minimum-error-probability (MEP) or maximum *a posteriori* probability (MAP) decision rule. What is the average error probability of the Bayesian decision rule? Is it smaller or larger than the average error probability of the maximum-likelihood decision rule? In the latter case, provide a brief explanation as to why the minimum-error-probability rule has a larger average error probability than another rule.

2. **[Detection problem for geometric random variables]**
A transmitter chooses one of two routes (Route 0 or Route 1) and repeatedly transmits a packet over the chosen route until the packet is received without error (that is, without CRC checksum failure) at the receiver. X denotes the number of times the packet is transmitted over the chosen route including the final error-free transmission. Assuming that the successive transmissions are independent trials of an experiment, the two hypotheses are

   - H_1: Route 1 is used for packet transmission: $X \sim \text{Geometric}(p_1)$
   - H_0: Route 0 is used for packet transmission: $X \sim \text{Geometric}(p_0)$

   where $0 < p_1 < p_0 < 1$ are the probabilities of error-free transmission over the two routes.

   (a) State the maximum-likelihood decision rule as to which route was used as a threshold test on the observed value of X.

   (b) Suppose the transmitter chooses Route 0 and Route 1 with probabilities π_0 and $\pi_1 = 1 - \pi_0$ respectively, i.e., π_0 and π_1 are the *a priori* probabilities of hypotheses H_0 and H_1. Assume that $0 < \pi_0 < 1$.

For what values of π_0 (if any) does the minimum-error-probability decision rule always choose hypothesis H_1 regardless of the value of the observation X?

For what values of π_0 (if any) does the minimum-error-probability decision rule always choose hypothesis H_0 regardless of the value of the observation X?

3. **[Conditional Probabilities]**

We say that event B gives *positive information* about event A if $P(A \mid B) > P(A)$, that is, the occurrence of B makes the occurrence of A more likely.

Now suppose that B gives positive information about A. If so

a) Does A give positive information about B?

b) Does B^c give *negative information* about A, that is, is it true that $P(A \mid B^c) < P(A)$?

c) Does B^c give positive information or negative information about A^c?

4. **[Bayes formula]**
The probability that a light bulb manufactured by Transylvania Corp. burns out during the n-th hour of operation is $p_1(n)$, $n = 1, 2, \ldots$. The probability that a light bulb manufactured by Eastinghouse Corp. burns out during the n-th hour of operation is $p_2(n)$, $n = 1, 2, \ldots$. Note that we are not giving you numerical values of these probabilities but do not forget that $\sum p_1(n) = \sum p_2(n) = 1$.

A bulb is equally likely to have been made by one of the two manufacturers.

Express your answers to the following questions in terms of $p_1(n)$ and $p_2(n)$.

a) What is the probability that the bulb burns out during the M-th hour of operation?

b) Given that the bulb burned out during the M-th hour of operation, what is the probability that it was manufactured by Transylvania Corp.?

c) Given that the bulb is still burning at the end of the M-th hour of operation, what is the probability that it was manufactured by Transylvania Corp.?

5. **[Bayes formula]**
Alice and Bob play the following game. First, Alice rolls a fair die and then Bob rolls the fair die. If Bob rolls a number at least as large as Alice's number, he wins the game. But if Bob rolled a number smaller than Alice's number, then Alice rolls the die again. If her second roll gives her a number that is less than or equal to Bob's number, the game ends with no winner (a tie, or draw as the British call it). If her second roll gives a number larger than Bob's number, Alice wins the game.

Find the probability that Alice wins the game and the probability that Bob wins the game. Also, find the probability of a tie directly (and not as $P(\text{tie}) = 1 - P(\text{Alice wins}) - P(\text{Bob wins})$.) If the three probabilities do not add up to 1, explain.

6. **[Bayes formula]**
Since there is no direct flight from San Diego (S) to New York (N), every time Alice wants to go from San Diego to New York, she has to connect through either Chicago (C) or Denver (D). There are flights every hour on the hour on the SC, SD, CN, and DN routes. Due to bad weather conditions, the SC flight can have a delay of one hour with probability p (and is on time with probability $1 - p$). If Alice's SC flight is late, she gets on the *next* CN flight (i.e., leaves an hour later than scheduled from Chicago) but the CN flight she takes also can have a delay of one hour with probability p, and independent of the delay or on-schedule arrival of the SC flight. A similar situation holds at Denver except that both incoming and outgoing flights are independently subject to *two* hour delays with probability q (and are on time with probability $1-q$), and thus if Alice's SD flight is late, she leaves Denver two hours behind schedule. Note that Alice has SuperPlatinum status on Unirican Airlines and is guaranteed to get on the next flight to New York even if someone else has to be bumped to make room for her.

Note: DO NOT assume that $q = 1 - p$ as in the notation often used in this course.

a) Find the probabilities that Alice arrives in N on schedule, an hour behind schedule, and two hours behind schedule if she chooses the SCN route. How late is she on average if she flies SCN?

b) Find the probabilities that Alice arrives in N on schedule, two hours behind schedule, and four hours behind schedule if she chooses the SDN route. How late is she on average if she flies SDN?

c) On average, how much behind schedule is Alice when she arrives in New York?

d) Suppose Alice arrives in New York two hours behind schedule. What is the probability that she flew the SCN route?

e) Suppose that Alice wants to maximize the probability that she arrives in New York less than 2 hours behind schedule. Under what conditions on p and q is the SCN route a better choice than the SDN route?

f) Suppose now that Alice always flies the SCN route. On average, how many trips does she make before experiencing a 2 hour delay?