University of Illinois **Spring 2011**

ECE 313: Problem Set 4
Confidence intervals, geometric distribution, ML parameter estimation, negative binomial distribution

Due: Wednesday February 16 at 4 p.m.
Reading: *313 Course Notes* Sections 2.5–2.8

1. **[Geometric Distribution]**
   Let X be a geometric random variable with parameter $1/4$, and let $Y = \sin(\pi X/2)$. Find the pmf of Y in closed-form.

2. **[Geometric Distribution]**
   An ECE senior student attends a career fair at the Illini Union. The probability p of receiving an interview request after a visiting a career fair booth depends on how well he/she did in ECE 313.

   (a) Let Y be random variable denoting the number of career fair booth visits a student must make before his/her first interview invitation. Find the pmf of Y.

   In particular, a C in 313 results in a probability $p_C = 0.15$ of an interview, whereas an A in 313 results in a probability of $p_A = 0.95$ of an interview.

   (b) On average, how many booth visits must a C student make before getting an interview?

   (c) On average, how many booth visits must an A student make before getting an interview? How does this compare to a C student?

   Now, suppose that there are a total of 5 booth visits that can be made.

   (d) Find the probability that an A student in 313 will not get an interview during the career fair.

   (e) Find the probability that a C student in 313 will get an interview during the career fair.

   (f) Find the minimum value of p for which a student can expect to get an interview by visiting five booths at the career fair. What does this mean, on average, about the C student in 313?

3. **[Confidence Intervals]**
   A communication system designer is simulating a communication link. The link is being designed for binary transmission, i.e., bits $b_k \in \{0,1\}$ are transmitted, and bits $\hat{b}_k \in \{0,1\}$ are received in the k^{th} bit-period. Noise and other sources of channel impairments result in *bit-errors*, i.e., the event $\hat{b}_k \neq b_k$. The bit errors are assumed to occur independently from one bit-period to the next with a probability of error or *bit error-rate* (BER) of p_e. The communication link is being designed to meet the specifications set by an international standards committee such as ITU, IEEE, or ETSI. The standards document specifies that $p_e \leq 10^{-4}$. The designer estimates p_e by running n bits through a simulation model, counting the number of errors E by comparing the transmitted bit b_k with recovered bit $\hat{b}_k$, and obtaining a BER estimate $\hat{p}_e = \frac{E}{n}$, where E is the error count, i.e., the total number of bits in error in a stream of n bits. The designer wants to impress the manager and wishes to report that the design meets the BER specifications with a $\pm 10\%$ tolerance around the 10^{-4} specified value.

   (a) Explain why E can be modeled as a binomial random variable with parameters (n, p_e).

   (b) What is the minimum value of n required to achieve a confidence level of 99%?

   (c) The designer has a desk-top with a dual core CPU running at a clock frequency of $4GHz$. The complexity of the simulation program and the associated compiler are such that each bit period takes 40 clock cycles (using both cores) to simulate. How much time in minutes will it take to simulate all n bits, where n is the answer to part (a).

4. **[Maximum-Likelihood Estimation and Confidence Intervals]**
An urn contains 11 red balls and an unknown number x of blue balls. The experiment consists of drawing one ball at random from the urn and noting its color. Consider 100 independent trials of this experiment. Thus, the ball drawn is replaced, and the urn shaken well before the next ball is drawn. It is observed that 20 of the drawings resulted in a red ball and 80 in a blue ball.

   (a) What is the maximum likelihood estimate $\hat{x}_{ML}$ of the number of blue balls x in the urn?

   (b) Let $\hat{p}_{ML} = \frac{11}{11+\hat{x}_{ML}}$. With what level of confidence can we say that $p = \frac{11}{11+x}$ lies in the interval $[\hat{p}_{ML} - 0.25, \hat{p}_{ML} + 0.25]$?

   (c) Find a confidence interval for p with confidence level 75%.

5. **[Maximum-likelihood Estimation]**
Let X denote a discrete random variable that takes on integer values $1, 2, \ldots, n$. The value of n is unknown, and we wish to find its maximum-likelihood estimate $\hat{n}_{ML}$ from the observation that X had value 11 on a trial of the experiment.

   (a) Explain why $\hat{n}_{ML}$ must be 11 or more.

   (b) Suppose that X has the increasing-ramp pmf $p_X(k) = \begin{cases} \frac{2k}{n(n+1)}, & 1 \leq k \leq n, \\ 0, & otherwise. \end{cases}$

   What is $\hat{n}_{ML}$ in this case?

   (c) Suppose that X has the decreasing-ramp pmf $p_X(k) = \begin{cases} \frac{2(n+1-k)}{n(n+1)}, & 1 \leq k \leq n, \\ 0, & otherwise. \end{cases}$

   Compute the value of $p_X(11)$ for $n = 11, 12, 13, \ldots$ to find the maximum-likelihood estimate $\hat{n}_{ML}$ numerically.

6. **[Negative Binomial Distribution]**
Let X be a random variable representing the number of times one must throw a red die until the outcome 2 has occurred four times. Let Y be a random variable representing the number of times one must throw a blue die until the outcomes 1 or 2 have occurred four times.

   (a) Find the pmf, the expected value and the standard deviation of X.

   (b) Find the pmf, the expected value and the standard deviation of Y.