University of Illinois Spring 2011

ECE 313: Problem Set 2

Due: Wednesday, February 3rd at 4 p.m.
Reading: Chapter I and II of the class notes

Grading policy: Please write all your derivations clearly - no partial credit will be given for solutions that do not include detailed derivations.

Please write your section number of on the top of your homework solution. You will be penalized 5 points if you fail to do so.

This problem set contains six problems involving counting and discrete random variables.

1. Suppose that an experiment is performed n times, where $n > 0$. For any event E of the sample space, let $n(E)$ denote the number of times that event E occurs, and define $f(E) = \frac{n(E)}{n}$. Show that $f(\cdot)$ satisfies Axioms 1, 2, and 3 for the probability function.

2. Of 30 students in a class,

   - 12 are not on Facebook
   - 9 are on both Facebook and Meetup
   - 2/3 of students not on Meetup don't have iPhones
   - At least one student is neither on Meetup nor on Facebook
   - 3 students on Facebook and having iPhones are not on Meetup
   - 2 students are on both Facebook and Meetup and have iPhones
   - 2/3 of the students not on Facebook and without iPhones are on Meetup

   Find how many students are not on Meetup, not on Facebook and don't have iPhones. Show your work. (Hint: Use a Karnaugh map. Fill in numbers or variables, trying to minimize the number of variables and equations needed.)

3. A bag contains n pairs of shoes in distinct styles and sizes. You pick two shoes at random from the bag. Note that this is sampling *without* replacement.

   a) What is the probability that you get a pair of matching shoes?

   b) What is the probability of getting one left shoe and one right shoe?

   Suppose now that $n \geq 2$ and that you choose 3 shoes at random from the bag.

   c) What is the probability that you have a pair of matching shoes among the three that you have picked?

   d) What is the probability that you picked at least one left shoe and at least one right shoe?

4. Suppose four people write their names on slips of paper; the slips of paper are randomly shuffled and then each person gets back one slip of paper; all possibilities of who gets

what slip are equally likely. Let X denote the number of people who get back the slip with their own name (i.e. the number of matches).

a) Provide a suitable sample space Ω to describe the experiment. How many elements does it have?

b) Find the PMF of X.

c) Find $E[X]$.

d) Find the probability that a given person gets her/his own name. Explain how this question is related to part (c).

e) Find $\mathrm{Var}(X)$.

5. Suppose that you are offered the following deal. You roll a fair die. If you roll a 6, you win 10 dollars. If you roll a 4 or 5, you win 8 dollars. If you roll a 1, 2, or 3, you pay 5 dollars.

   a) What are you ultimately interested in here (the value of the roll or the money you win)?

   b) In words, define the Random Variable X that describes this experiment.

   c) List the values that X may take on.

   d) Describe its PMF.

   e) Over the long run of playing this game, what are your expected average winnings per game?

   f) Based on numerical values, should you accept this deal? Explain your decision in complete sentences.

6. Let $n \geq 1$ be an integer and suppose the random variable X has the PMF

$$p_X(k) = \begin{cases} \frac{2k}{n(n+1)} & 1 \leq k \leq n \\ 0 & \text{else.} \end{cases}$$

   (a) Verify that p_X is a valid PMF.
   (b) Find the mean and variance of X.
   (c) Find the PMF of $Y = 1/X$.
   (d) Find the expected value of Y directly and using LOTUS.