University of Illinois Spring 2011

ECE 313: Midterm Exam I

Monday February 28, 2011
7:00 p.m. — 8:15 p.m.
1NHB 228 in Natural History Building

1. **[25 points]** Bob flips two fair coins. Let N be the number of heads that are showing. Mary now draws N cards with replacement from a fair deck of 52 cards (i.e., Mary draws one card, notes its suit, replaces the card in the deck, and repeats this process N times). Let M be the number of clubs that she draws.

   (a) **[6 points]** Compute the probability that $M = 2$.
   Solution: For this question, you'll need to use total probability:

   $$\begin{aligned} P(M=2) &= P(M=2 \mid N=0)P(N=0) + \\ &\quad P(M=2 \mid N=1)P(N=1) + \\ &\quad P(M=2 \mid N=2)P(N=2) \end{aligned}$$

   This isn't so bad, since $P(M = 2 \mid N = 0) = P(M = 2 \mid N = 1) = 0$ (you cannot draw two clubs unless you have the right to draw at least two cards). Now things become easier, $P(M = 2 \mid N = 2)$ is simply the binomial distribution with $N = 2$ and $p = 1/4$ (note, sampling with replacement means that the probability of drawing a club is $1/4$ for both draws, and the two draws are independent). Thus,

   $$P(M=2 \mid N=2) = \binom{2}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^0 = 1/16$$

   For $P(N = 2)$, since the coin flips are independent, this is merely the product $(1/2)(1/2) = 1/4$. Put all of this together, and we find that $P(M = 2) = 1/64$.

   (b) **[5 points]** Suppose Mary draws 2 clubs. Compute the probability that N was 2 (i.e., compute $P\{N = 2 \mid M = 2\}$).
   Solution: There are two ways to solve this problem. The first way is to merely note that Mary cannot draw two clubs unless she draws two cards. Therefore, if she draws two clubs, she *must* have drawn two cards, and $P\{N = 2 \mid M = 2\} = 1$. The second way to solve this is to use the cel

   $$P\{N=2 \mid M=2\} = \frac{P\{M=2 \mid N=2\}P\{N=2\}}{P\{M=2\}} = \frac{(1/16)(1/4)}{1/64} = 1$$

   All of the individual pieces were solved above in Part a.

   (c) **[6 points]** Suppose now that two heads are shown. Compute the probability that one club is drawn (i.e., compute $P\{M = 1 \mid N = 2\}$).
   Solution: Given that $N = 2$, the conditional probability is, again, simply the binomial distribution with parameters $p = 1/4$ and $N = 2$, and with one successful outcome (i.e., $k = 1$ in the usual notation)

   $$P(M=1 \mid N=2) = \binom{2}{1}\left(\frac{1}{4}\right)^1\left(\frac{3}{4}\right)^1 = 6/16 = 3/8$$

(d) **[8 points]** You are told by a savvy ECE 313 student that $P\{M = 0\} = \frac{49}{64}$. Compute the conditional probability $P\{N = 2 \mid M = 1\}$.

Solution: The easiest way to solve this is to again use the theorem of Bayes:

$$P\{N = 2 \mid M = 1\} = \frac{P\{M = 1 \mid N = 2\}P\{N = 2\}}{P\{M = 1\}} = \frac{(3/8)(1/4)}{7/32} = 3/7$$

The pieces in the numerator were computed in parts (a) and (c). For $P\{M = 1\}$, one need merely recall that $P\{M = 0\} + P\{M = 1\} + P\{M = 2\} = 1$, since M can only take these three values. You were given $P\{M = 0\}$, and you computed $P\{M = 2\}$ in part (a), so it is a simple matter of subtraction to compute $P\{M = 1\}$.

2. **[25 points]** In this problem, you are going to encounter the binomial, geometric and Poisson random variable.

(a) **[10 points]** Let X be a binomial random variable with parameters $n = 10,000$ and $p = 0.0001$. *Approximately* compute the numerical value for the probability $P\{X = 3\}$ (feel free to use the approximation $e \simeq 3$). Let Y be another binomial random variable with parameters $n = 10$ and $p = 0.2$. Compute $P\{Y < 3\}$. Justify your solutions.

Solution: For the first part, use the Poisson approximation for a binomial random variable: $\lambda = np = 1$, and $P\{X = 3\} \simeq 3^{-1}1^3/(3!) = 1/18$.
For the second part, notice that you cannot use the Poisson approximation - you have to compute

$$P\{Y < 3\} = P\{Y = 0\} + P\{Y = 1\} + P\{Y = 2\}, \ \ P\{Y = k\} = \binom{n}{k} p^k \, (1-p)^k, \quad (1)$$

where $n = 10$ and $p = 0.2$.

(b) **[10 points]** Let $Y = X^2$ with X being a Poisson random variable that has parameter λ, and let $Z = 2W + 3$, where W is a geometric random variable with parameter p. Find the expected value of Y, and the expected value and variance of Z.

Solution: We know that $var(X) = E[X^2] - E[X]^2$. Since the variance and the mean of a Poisson RV are both equal to λ, we have $E[Y] = \lambda + \lambda^2$.
We also know that $E[aW + b] = aE[W] + b$ and that $var(aW + b) = a^2\, var(W)$. This gives $E[Z] = 2/p + 3$ and $var(Z) = 4\,(1-p)/p^2$.

(c) **[5 points]** Find the maximum likelihood estimator for the variance of a Poisson random variable, based on one single experiment. Justify your answer.

Solution: The expected value and variance of a Poisson random variable are equal. Hence, a ML estimator for λ will do the job - and it was proved in the lecture notes that $\lambda_{ML} = k$, where k is the outcome (observation) of the single experiment performed.

3. **[25 points]** In this problem, you are going to address some counting problems.

(a) **[15 points]** Find the number of ways four indistinguishable electrons can occupy six orbits so that no orbit contains more than two electrons.

Solution: We can have two orbits with two electrons each, or three orbits occupied, one with two and two with one electron, or four orbits containing exactly one electron. In the first case, we have $\binom{6}{2}$ choices, in the second case we have $\binom{6}{1}\binom{5}{2}$ choices, while in the third case we have exactly $\binom{6}{4}$ choices. This gives $15 + 60 + 15 = 90$ choices. Note that I also accepted the solution $\binom{6}{2} + \binom{6}{3} + \binom{6}{4} = 50$, if you assume that the orbits are indistinguishable as well.

(b) **[10 points]** Assume that you have four electrons, half of which have spin equal to $+1/2$ and half of which have spin equal to $-1/2$. Find the number of ways to place the electrons on ten orbits so that each orbit has total spin sum zero.

Solution: Each orbit has to contain either zero or two or four electrons. Hence, the solution is just $\binom{10}{2} + \binom{10}{1} = 55$. Notice that if you assumed that the electrons obey the constraint of problem part a) (which was not part of the statement), you would have obtained $\binom{10}{2} = 45$. I deducted only one point in this case.

4. **[25 points]** Let X denote a discrete random variable that takes on integer values $0, 1, 2, \ldots, n$, with pmf $p_X(k) = \begin{cases} \frac{a^k(1-a)}{1-a^{n+1}}, & 0 \leq k \leq n, \\ 0, & otherwise, \end{cases}$
for $a \in (0, 1) \cup (1, \infty)$. The value of n is unknown, but it is known that for physical reasons it cannot be greater than 50.

(a) **[8 points]** Assuming that $a = 1/2$, find the maximum-likelihood estimate $\hat{n}_{ML}$ from the observation that $X = 15$ on a trial of the experiment.

Solution: The observation is $X = 15$, so $k = 15$. The maximum-likelihood estimate $\hat{n}_{ML}$ is therefore the value of n that maximizes $p_X(15) = \frac{(1/2)^{15}(1-1/2)}{1-(1/2)^{n+1}} = \frac{(1/2)^{16}}{1-(1/2)^{n+1}}$.
Notice that as n increases, $(1/2)^{n+1}$ decreases, so $1-(1/2)^{n+1}$ increases and hence $p_X(15)$ decreases. So we need to choose the smallest possible $n \geq 0$, which is $n = 15$ because if we choose $n < 15$ then we could have not observed $X = 15$. Therefore, $\hat{n}_{ML}$=15.

(b) **[8 points]** Assuming that $a = 3/2$, find the maximum-likelihood estimate $\hat{n}_{ML}$ from the observation that $X = 15$ on a trial of the experiment.

Solution: Similarly to part (a), we want to find the value of n that maximizes $p_X(15) = \frac{(3/2)^{15}(1-3/2)}{1-(3/2)^{n+1}} = \frac{(1/2)(3/2)^{15}}{(3/2)^{n+1}-1}$.
Notice that as n increases, $(3/2)^{n+1}$ increases, so $(3/2)^{n+1}-1$ increases and hence $p_X(15)$ decreases. So we need to choose the smallest possible $n \geq 0$, which is again $\hat{n}_{ML}$=15.

(c) **[4 points]** In the case when $a = 1$ the pmf of X given above is replaced by $p_X(k) = c$ for $0 \leq k \leq n$, and zero otherwise. Find the constant c that makes this a valid pmf.

Solution: For it to be a valid pmf we need:
$1 = \sum_{k=0}^{n} p_X(k) = \sum_{k=0}^{n} c = c(n+1) \Rightarrow c = \frac{1}{n+1}$.
Note that $k = 0$ is included so the sum includes $n + 1$ terms, not n.

(d) **[5 points]** If the pmf of X is the one given in part (c), find the expected value of X.

Solution: $E[X] = \sum_{k=0}^{n} kp_X(k) = \sum_{k=0}^{n} kc = c\sum_{k=0}^{n} k = c\ \frac{n(n+1)}{2} = \frac{n(n+1)}{2(n+1)} = \frac{n}{2}$.