University of Illinois **Spring 2010**

ECE 313: Problem Set 5
Conditional Probability, Law of Total Probability, Bayes' Formula

Due:	Wednesday February 24 at 4 p.m.
Reading:	*Ross* Chapter 3; Powerpoint Lecture Slides, Sets 9-14
Noncredit Exercises:	**Chapter 3:** Problems 1, 2, 5, 10, 12, 16, 31, 38, 39, 51, 52 Theoretical Exercises 1, 2, 8, 16; Self-Test Problems 1-14.
Reminders:	Exam February 25, No class on Friday February 26 but office hours as usual next week

1. [**Definition of Conditional Probability**]
60% of all apples in the supermarket are red. 30% of the produce in the supermarket is red, and 50% of the red produce items are apples. What fraction of all produce items in the supermarket are apples?

2. [**Conditional Probability and Total Probability**]
Simplify the following expresions.

   (a)
$$P(A \cup B \cup C | BC)$$

   (b)
$$P(BC | B^C \cup C^C)$$

   (c)
$$P(ABCD|E) + P(A^C BCD|E)$$

   (d)
$$P(A|BC)P(B|C)P(C) + P(A|B^C C)P(B^C|C)P(C)$$

3. [**Bernoulli Bus Lines**]
Once per minute, a bus arrives at Sixth and Green with probability P. Consecutive minutes are independent; knowing that a bus arrived at 4:37 tells you nothing about what will happen at 4:38. Buses arriving after 5:00PM always arrive in the following order: the Turquoise line (#16) always comes first, followed by the Fuchsia line (#102), followed by the Octarine (#42). Your bus is the Octarine.

   (a) You arrive at the bust stop at exactly 5:00.
      i. What is the probability that you will wait exactly k minutes for your bus?
      ii. What is your expected waiting time?
      iii. What is the variance of your waiting time?

   (b) Your 4:00 lecturer (being half elvish) has a poor grasp of time, therefore you reach the bus stop at 5:05PM. What is the probability that your bus arrives m minutes after you? Note that m may be negative.

   (c) Fortunately, your friend Joe has been at the bus stop since 5:00. He says you haven't missed your bus; the Turquoise bus came at 5:03, but the Fuchsia and Octarine buses have not arrived yet. Assuming that Joe is reliable, what is the probability that you will wait m minutes for your bus?

   (d) Aja has also been at the bus stop since 5:00, but she disagrees with Joe. She believes that both the Turquoise and Fuchsia buses have come and gone, but that the Octarine bus has not yet arrived. Based on your knowledge of their honesty, reliability, and observational skills, you suppose that Aja is correct with probability 0.75, and Joe is correct with probability 0.25.

i. What is the probability that you will wait m minutes for your bus to arrive?
ii. What is your expected waiting time?
iii. What is the variance of your waiting time?

4. **[Epidemiology 101]**
It is estimated that as many as 300,000 Americans are currently pre-symptomatic carriers of the Romulan Retrovirus. Curing the disease after a patient has become symptomatic costs \$5000. Curing the disease *before* a patient becomes symptomatic can be accomplished using a Vulcan mind meld administered over the internet, at a cost of only \$50 per person. In order to know whether or not a patient has the disease, it is necessary to administer a 10-cent diagnostic test. The test returns a red flag 99.8% of the time if the patient is infected ($P(\text{red}|H_1) = 0.998$, where H_1 is the hypothesis that the patient is infected), and a blue flag otherwise. If a patient is not infected, the test returns a blue flag 99.8% of the time ($P(\text{blue}|H_0) = 1 - P(\text{red}|H_0) = 0.998$).

   (a) Bob Skinnyknees is tested for the virus, and the test comes back with a red flag. What's the probability that Bob is infected? You may assume that there are 300 million people in the United States.

   (b) The Surgeon General of the United States proposes testing all Americans, and administering the Vulcan mind meld to those who test positive. What is the expected cost of this strategy? Be sure to include the cost of the test itself, the expected cost of the mind melds, and the expected cost of applying the \$5000 post-symptomatic treatment to patients whose disease is not detected by the diagnostic test.

   (c) The Surgeon Specific of Romulan Retroviruses proposes a different strategy. She proposes that Americans who test positive once should be tested a second time. Only those who get two red flags in a row will be recommended for the Vulcan mind meld. You may assume that if a person is infected, the test results are independent of one another, i.e., $P(\text{red}, \text{red}|H_1) = P(\text{red}|H_1)^2$, and likewise for H_0. What is the expected cost of this strategy?

5. **[Hold or Trade?]**
An investment adviser offers you two portfolios. One portfolio is known to increase in value on 70% of all trading days, and to decrease in value on 30% of all days. The other portfolio increases in value only 40% of the time, and decreases otherwise. Unfortunately, you don't know which is which, and your so-called adviser refuses to tell you, so you decide on the following strategy. You will choose a portfolio at random (by flipping a fair coin), and hold it for one day. If it increases in value on the first day, you will continue to hold it; if not, you will sell it, and buy the other portfolio instead. Using this strategy, what is the probability that your portfolio increases in value on the second day?

6. **[Polya's urn model: a classic result in probability theory]**
An urn contains r red and g green balls. Two balls are drawn at random from the urn, with the first ball being returned to the urn (which is then shaken well, not stirred) before the second ball is drawn. Let R_1 and R_2 respectively denote the events that the first and second balls are red.

   (a) What are the values of $P(R_1)$ and $P(R_2)$?

   (b) Now suppose that when the first ball is returned to the urn, c *additional* balls of the *same color* are also put into the urn (which is then shaken well before the second ball is drawn.) Clearly $P(R_1)$ is the same as before, but what is $P(R_2)$ now? Remember that the urn now contains $r + g + c$ balls. Simplify your answer and compare to the value of $P(R_2)$ that you obtained in part (a).

   (c) For the experiment of part (b), what is the conditional probability that the urn contained $r + c$ red balls given that R_2 occurred?

7. **[The (in)famous Monty Hall problem]**
Monty Hall, the host of the TV game show "Let's Make A Deal" shows you three curtains. One curtain conceals a car, while the other two conceal goats. All three curtains are equally likely to conceal the car. He offers you the following "deal": pick a curtain, and you can have whatever is behind it. When you pick a curtain, instead of giving you your just deserts, Monty (who knows where the car is) opens

one of the remaining curtains to show you that there is a goat behind it, and offers the following "new, improved deal": you can either stick with your original choice, or switch to the remaining (unopened) curtain. Amidst the deafening roars of "Stand pat!" and "Switch, you idiot!" from the crowd, Monty points out that previously your chances of winning were 1/3. Now, since you know that the car is behind one of the two unopened curtains, your chances of winning have increased to 1/2, and thus the new improved deal is indeed better.

(a) Let A denote the event that your first choice of door has the car behind it. What is $P(A)$?

Let B denote the event that your second choice of door has the car behind it.

(b) *Your strategy is to always stay put* and so your second choice is the same as your first choice. What is $P(B \mid A)$ in this case? What is $P(B \mid A^c)$? Use these results to find $P(B)$ for the stay-put strategy.

(c) *Your strategy is to always switch* and so your second choice is the other unopened door. What is $P(B \mid A)$ in this case? What is $P(B \mid A^c)$? Use these results to find $P(B)$ for the always-switch strategy.

(d) *Your strategy is to pick randomly* and so your second choice is equally likely to be either unopened door. What is $P(B \mid A)$ in this case? What is $P(B \mid A^c)$? Use these results to find $P(B)$ for the pick-randomly strategy. Is Monty correct in asserting that if you choose randomly between the two unopened curtains, you have a probability of winning of 1/2?

(e) Having disposed of your goat, you return the next day to the show, and this time, Monty calls you *and* your friend to come on down and choose one curtain each. Which is better: to be the first to pick a curtain or the second? Or does it not make a difference? This time, Monty opens the curtain chosen by your friend to reveal a goat and sends him back to his seat. He now asks whether you want to stick with your original choice or switch to the the third (unchosen) curtain. Which choice gives you a larger chance of winning the car?

Note: The rules of the game of parts (a)-(d) are that Monty, who knows which curtain conceals the car, always opens one of the two unchosen curtains and he always offers the "new improved deal," that is, he never opens a curtain to reveal the prize (saying "Oops, you lose; return to your seat."). In the game of part (e), he always opens one of the chosen curtains to eliminate one of the contestants and then always offers the other contestant the chance to switch.