University of Illinois Spring 2010

ECE 313: Problem Set 6

Due: Wednesday, March 3rd at 4 p.m.
Reading: Ross, Chapter 3; Lecture Notes 14-18.
Noncredit Exercises: DO NOT turn these in.
Chapter 3: Problems 80,84 and 86;
Theoretical Exercises 8,16 and 21.

This Problem Set contains seven problems.

1. [**Conditional Probability**]
Die A has four red and two white faces, whereas die B has two red and four white faces. A fair coin is flipped once. If it falls heads, the game continues by throwing die A alone. If it falls tails, die B is to be used.

   (a) Show that the probability of red at any throw is $1/2$.

   (b) If the first throw resulted in red, what is the probability of red in the third throw?

   (c) If red turns up in the first n throws, what is the probability that die A is being used?

   Solution: Let H be the event that the fair coin lands heads. Then $P(H) = P(H^c) = 1/2$. In addition, let R denote the event that the outcome of rolling a die is red.

   (a) The law of total probability asserts that $P(R) = P(H)P(R|H) + P(H^c)P(R|H^c) = 1/2$.

   (b) Let R_1 be the event that the first throw results in red, and similarly, let R_3 be the event that the third throw results in red. Then $P(R_3|R_1) = \frac{P(R_1R_3)}{P(R1)} = \frac{\frac{1}{2}\cdot(\frac{2}{3})^2+\frac{1}{2}\cdot(\frac{1}{3})^2}{\frac{1}{2}} = \frac{5}{9}$.

   (c) If N denotes the event that red shows up in the first n throws, then Bayes formula gives
$$P(A|N) = \frac{P(A)P(A|N)}{P(N)} = \frac{(1/2)\,(2/3)^n}{1/2((1/3)^n + (2/3)^n)} = \frac{2^n}{1+2^n}.$$

2. [**Conditional Probability and Poisson Random Variables**]
Each customer who enters Laura's clothing store will purchase a suit with probability p. If the number of customers entering the store is Poisson distributed with mean λ, what is the probability that Laura does not sell any suits? What is the probability of her selling k suits?

   Solution: Let X be the number of suits that Laura sells, and let N denote the number of customers who enter the store. By conditioning on N we see that
$$P\{X = 0\} = \sum_{n=0}^{\infty} P\{X = 0|N = n\}P\{N = n\} = \sum_{n=0}^{\infty} (1-p)^n\,\frac{e^{-\lambda}\lambda^n}{n!} = e^{-\lambda\,p}.$$

Similarly, by observing that

$$P\{X = k|N = n\} = \binom{n}{k} p^k (1-p)^{n-k}, \text{ for } n \geq k,$$

and equals zero otherwise, one obtains

$$P\{X = k\} = e^{-\lambda p} \frac{(\lambda p)^k}{k!}.$$

3. **[Law of total probability]**
Alice and Bob play the following game. First, Alice rolls a fair die and then Bob rolls the fair die. If Bob rolls a number at least as large as Alice's number, he wins the game. But if Bob rolled a number smaller than Alice's number, then Alice rolls the die again. If her second roll gives her a number that is less than or equal to Bob's number, the game ends with no winner (a tie, or draw as the British call it). If her second roll gives a number larger than Bob's number, Alice wins the game.

Find the probability that Alice wins the game and the probability that Bob wins the game. Also, find the probability of a tie directly (and not as $P(\text{tie}) = 1 - P(\text{Alice wins}) - P(\text{Bob wins})$.) If the three probabilities do not add up to 1, explain.

Solution: Let us define the random variable of the face that Bob's roll of dice shows by b. Let B denote the event that Bob wins the game. Then, since Alice can lose i different ways if $b = i$, it is clear that for $i = 1, 2, 3, 4, 5$, and 6, $P(B \mid b = i) = i/6$. By the law of total probability,

$$P(B) = \sum_{i=1}^{6} P(B \mid b = i) P(b = i) = \frac{1+2+3+4+5+6}{6} \times \frac{1}{6} = \frac{21}{36} = \frac{7}{12}.$$

Next, note that *both* of Alice's rolls must result in numbers *larger* than Bob's number in order for Alice to win. Let A denote the probability that Alice wins the game. If Bob rolls an i, then the (conditional) probability that Alice beats him twice is $P(A \mid b = i) = \frac{(6-i)^2}{36}$. Again by the law of total probability,

$$P(A) = \sum_{i=1}^{6} P(A \mid b = i) P(b = i) = \frac{5^2 + 4^2 + 3^2 + 2^1 + 1^2 + 0^2}{36} \times \frac{1}{6} = \frac{55}{216}.$$

Let D denote the event of a draw, which occurs whenever Alice's first roll beats Bob, but the second does not. Thus, $P(D \mid b = i) = \frac{(6-i)i}{36}$. Once again, the law of total probability gives us

$$P(D) = \sum_{i=1}^{6} P(D \mid b = i) P(b = i) = \frac{5 \times 1 + 4 \times 2 + 3 \times 3 + 2 \times 4 + 1 \times 5 + 0 \times 6}{36} \times \frac{1}{6}$$

which equals $\frac{35}{216}$.

Sure, the three probabilities add up to 1 (either Bob wins, or Alice wins or there is a tie): no surprise here, you can verify this directly by summing up the three probabilities.

4. **[Conditional Expectations]**
A miner is trapped in a mine containing three doors. The first door leads to a tunnel that takes him to safety after two hours of travel. The second door leads to a tunnel that takes him to the mine after three hours of travel. The third door leads into a tunnel that returns him to his mine after five hours. Assuming that the miner is at all times equally likely to choose any of the doors, what is the expected length of time until the miner reaches safety?

Solution: Let x denote the time until the miner reaches safety, and let Y denote the door he picked initially. Then

$$E[X] = E[X|Y=1]P\{Y=1\} + E[X|Y=2]P\{Y=2\} + E[X|Y=3]P\{Y=3\}$$

and consequently

$$E[X] = \frac{1}{3}[E[X|Y=1] + \frac{1}{3}E[X|Y=2] + \frac{1}{3}E[X|Y=3]].$$

However,

$$E[X|Y=1] = 2, E[X|Y=2] = E[X]+3, E[X|Y=2] = E[X]+5,$$

which gives $E[X] = 10$.

5. **[Conditional Variance]**
Let X be a random variable with mean μ and variance σ^2. Let F be an even of positive probability. Express the conditional variance of $Y = (3X-\mu)/(2\sigma)$ given F in terms of the conditional variance of X given F.

Solution: Since $var(Y|F) = E[Y^2|F] - E[Y|F]^2$, and since expectation is a linear operator, it follows that

$$var(Y|X) = \frac{9}{4\sigma^2} var(X|F).$$

6. **[Decision Making]**
If H_0 is the true hypothesis, the random variable X takes on values 0, 1, 2, and 3 with probabilities 0.1, 0.2, 0.3, and 0.4 respectively. If H_1 is the true hypothesis, the random variable X takes on values 0, 1, 2, and 3 with probabilities 0.4, 0.3, 0.2, and 0.1 respectively.

   (a) Write down the likelihood matrix L and indicate the *maximum-likelihood decision rule* by shading the appropriate entries in L. What is the false-alarm probability P_{FA} and what is the missed-detection probability P_{MD} for the maximum-likelihood decision rule?

   Solution: The likelihood matrix L is as shown below and the maximum-likelihood decision rule is indicated shading.

Hypothesis	$X=0$	$X=1$	$X=2$	$X=3$
H_1	0.4	0.3	0.2	0.1
H_0	0.1	0.2	0.3	0.4

It is easy to get $P_{\text{FA}} =$ sum of unshaded entries on H_0 row $= 0.1 + 0.2 = 0.3$
and $P_{\text{MD}} =$ sum of unshaded entries on H_1 row $= 0.1 + 0.2 = 0.3$ also.

(b) Suppose that the hypotheses have *a priori* probabilities $\pi_0 = 0.7$ and $\pi_1 = 0.3$. Use the law of total probability to find the average error probability of the maximum-likelihood decision rule that you found in part (a).

Solution: By the law of total probability, $P(E) = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = 0.7 \times 0.3 + 0.3 \times 0.3 = 0.3$.

(c) Use the *a priori* probabilities given in part (b) to find the joint probability matrix J and indicate on it the Bayesian decision rule, which is also known as the minimum-error-probability (MEP) or maximum *a posteriori* probability (MAP) decision rule. What is the average error probability of the Bayesian decision rule? Is it smaller or larger than the average error probability of the maximum-likelihood decision rule? In the latter case, provide a brief explanation as to why the minimum-error-probability rule has a larger average error probability than another rule.

7. **[Detection problem for geometric random variables]**
A transmitter chooses one of two routes (Route 0 or Route 1) and repeatedly transmits a packet over the chosen route until the packet is received without error (that is, without CRC checksum failure) at the receiver. X denotes the number of times the packet is transmitted over the chosen route including the final error-free transmission. Assuming that the successive transmissions are independent trials of an experiment, the two hypotheses are

- H_1: Route 1 is used for packet transmission: $X \sim \text{Geometric}(p_1)$
- H_0: Route 0 is used for packet transmission: $X \sim \text{Geometric}(p_0)$

where $0 < p_1 < p_0 < 1$ are the probabilities of error-free transmission over the two routes.

(a) State the maximum-likelihood decision rule as to which route was used as a threshold test on the observed value of X.

Solution: If $X = n$, the likelihood ratio has value

$$\Lambda(n) = \frac{p_1(1-p_1)^{n-1}}{p_0(1-p_0)^{n-1}} = \frac{p_1}{p_0}\left(\frac{1-p_1}{1-p_0}\right)^{n-1} > 1 \text{ if } (n-1)\ln\left(\frac{1-p_1}{1-p_0}\right) > \ln\left(\frac{p_0}{p_1}\right)$$

Since $p_1 < p_0$, we have that $1-p_1 > 1-p_0$ and $\ln((1-p_1)/(1-p_0)) > 0$. Therefore, the maximum likelihood decision rule is

$$\text{“Decide that } H_1 \text{ is the true hypothesis if } X > 1 + \frac{\ln\left(\frac{p_0}{p_1}\right)}{\ln\left(\frac{1-p_1}{1-p_0}\right)}.\text{”}$$

Note that there is less chance of a successful transmission on Route 1 than on Route 0, and hence large values of X *should* lead to the decision that Route 1 was used.

(b) Suppose the transmitter chooses Route 0 and Route 1 with probabilities π_0 and $\pi_1 = 1 - \pi_0$ respectively, i.e., π_0 and π_1 are the *a priori* probabilities of hypotheses H_0 and H_1. Assume that $0 < \pi_0 < 1$.

For what values of π_0 (if any) does the minimum-error-probability decision rule always choose hypothesis H_1 regardless of the value of the observation X?

For what values of π_0 (if any) does the minimum-error-probability decision rule always choose hypothesis H_0 regardless of the value of the observation X?

Solution: The minimum-error-probability (MEP) decision rule decides that H_1 is the true hypothesis if the likelihood ratio exceeds the threshold π_0/π_1. Now $\Lambda(1) = p_1/p_0 < 1$. Since $1 - p_1 > 1 - p_0$, we see that

$$\Lambda(n) = \frac{p_1(1-p_1)^{n-1}}{p_0(1-p_0)^{n-1}} = \frac{p_1(1-p_1)^{n-2}}{p_0(1-p_0)^{n-2}}\left(\frac{1-p_1}{1-p_0}\right) = \Lambda(n-1)\left(\frac{1-p_1}{1-p_0}\right) > \Lambda(n-1),$$

and so $\Lambda(1) = p_1/p_0$ is the smallest value of the likelihood ratio. It follows that if $\pi_0/\pi_1 = \pi_0/(1-\pi_0) < p_1/p_0$, that is, if $\pi_0 < p_1/(p_0+p_1)$, the MEP decision rule will always decide that H_1 is the true hypothesis regardless of the observed value of X.

On the other hand, since $\Lambda(n)$ increases monotonically without bound as n increases, there is *no* value of $\pi_0 < 1$ for which π_0/π_1 can be guaranteed to be larger than the likelihood ratio no matter what value X takes on.