University of Illinois Spring 2010

ECE 313: Hour Exam I

Thursday February 25, 2010
7:00 p.m. — 8:00 p.m.

Name: **(in BLOCK CAPITALS)** ____

University ID Number: ____________________________

Signature: ____

Section: ☐ C, 10:00 a.m. ☐ D, 11:00 a.m. ☐ F, 12:00 p.m. ☐ E, 1:00 p.m.

Instructions

This exam is closed book and closed notes except that one 8.5”×11” sheet of notes is permitted: both sides may be used. No electronic devices whatsoever are allowed.

The exam consists of four problems worth a total of 100 points. The problems are not weighted equally, so it is best for you to pace yourself accordingly. Write your answers in the spaces provided, and reduce common fractions to lowest terms, but do not convert them to decimal fractions (for example, write $\frac{3}{4}$ instead of $\frac{24}{32}$ or 0.75).

SHOW YOUR WORK. Answers without justification will receive no credit. If you need extra space, use the back of the previous page.

Grading	
1. 20 points	____________
2. 20 points	____________
3. 30 points	____________
4. 30 points	____________
Total (100 points)	____________

1. **[20 points]** Consider two events, A and B. You are told that $P(A) = 0.55$ and that $P(B) = 0.82$.

   (a) **[5 points]** What is the maximum possible value for $P(A \cap B)$? Justify. **Solution:** $P(A \cap B)$ is maximized when $A \subset B$, and the maximum value of the probability is therefore equal to 0.55.

   (b) **[5 points]** What is the minimum possible value for $P(A \cap B)$? Justify.
   Solution: Note that A and B cannot be disjoint, since $0.55 + 0.82 = 1.37$. Hence, the minimum value for $P(A \cap B)$ equals $P(A) + P(B) - 1 = 0.37$.

   (c) **[5 points]** What is the maximum possible value for $P(A \cup B)$? Justify.
   Solution: $P(A \cup B)$ cannot exceed the value 1 - this is the maximum of $P(A \cup B)$.

   (d) **[5 points]** What is the minimum possible value for $P(A \cup B)$? Justify.
   Solution: Since $P(B) = 0.82$, the minimum possible value for $P(A \cup B)$ cannot be smaller than 0.82.

2. **[20 points]** The pmf of a discrete RV is known and given by

$$\begin{aligned} p_X(-1) &= \frac{b^2+2}{3} - b, \\ p_X(0) &= 2b^2, \\ p_X(+1) &= \frac{2b-b^2}{3}, \\ p_X(u) &= 0, \text{ for all other } u. \end{aligned} \tag{1}$$

Find the numerical value for $p_X(0)$. Is this answer unique? Justify your solution. **Solution:** We must have $p_X(0) + p_X(-1) + p_X(+1) = 1$, which gives $\frac{6b^2-b+2}{3} = 1$. Consequently,

$$b = \frac{1}{12} \pm \frac{5}{12}$$

but choosing $b = -1/3$ yields $p_X(1) < 0$ and $p_X(-1) > 1$, therefore $b = 1/2$ is the unique correct solution.

3. **[30 points]**

(a) **[10 points]**
Passenger trains are constructed as follows. The passenger train company owns 20 train cars. If a car is functional, it is added to the train. The probability that a car is functional, on any given day, is $P_f = 0.9$. Let $p_X(k|\text{passenger})$ denote the conditional probability that the length of the train (X) equals any particular number of cars (k), given that the train is a passenger train. What is $p_X(k|\text{passenger})$? Your answer should be a function of k; you may also express it as a function of P_f if you wish. **Solution:**

$$p_X(k|\text{passenger}) = \binom{20}{k} P_f^k (1 - P_f)^{20-k}$$

(b) **[10 points]**
Freight trains are constructed as follows. Four shipping companies each take turns adding cars to the train. The first company's foreman, Alvin, attaches cars to the train until he falls asleep. Alvin falls asleep after any given train car with probability $P_s = 0.1$. After Alvin finishes, the second company's foreman, Bob, begins attaching cars to the train; after any given train car, Bob falls asleep (and stops attaching cars) with probability $P_s = 0.1$. After Bob finishes, then Carl gets to attach cars; like the others, Carl falls asleep with probability $P_s = 0.1$ after attaching any particular car. Finally, after Carl is finished, Doug begins attaching cars; like the others, Doug falls asleep after any given car with probability $P_s = 0.1$. When Doug falls asleep, the engineer drives the train to Chicago. Let $p_X(k|\text{freight})$ denote the conditional probability that the length of the train (X) equals any particular number of cars (k), given that the train is a freight train. What is $p_X(k|\text{freight})$? Your answer should be a function of k; you may also express it as a function of P_s if you wish. **Solution:**

$$p_X(k|\text{freight}) = \binom{k-1}{3} P_s^4 (1 - P_s)^{k-4}$$

(c) **[5 points]**
Ten percent of Illinois trains are passenger trains; ninety percent are freight trains. There is a train of some kind going through Champaign right now. What is $p_X(k)$? You may express your answer in terms of $p_X(k|\text{passenger})$ and $p_X(k|\text{freight})$. **Solution:**

$$p_X(k) = 0.1p_X(k|\text{passenger}) + 0.9p_X(k|\text{freight})$$

(d) **[5 points]** Ten percent of Illinois trains are passenger trains; ninety percent are freight trains. There is a train of some kind going through Champaign right now. What is $E[X]$, its total expected length? (numerical answer) **Solution:**

$$E[X] = 0.1 \cdot 20 \cdot 0.9 + 0.9 \cdot 4 \cdot 10 = 37.8$$

4. **[30 points]** Consider an earthquake-prone town where the number of earthquakes that occur in intervals of time Δ days are independent, and where Δ is assumed to be very small. The probability of an earthquake occurring in an interval of time Δ is given by $\lambda\Delta$, for some $\lambda > 0$.
Hint: At this point, you may want to recall the lecture pertaining to the relationship between Poisson and Binomial random variables

   (a) **[10 points]** Calculate the probability that an earthquake does not occur in 10 days.
   Solution: If we split 10 days into intervals of length Δ, then there are $\frac{10}{\Delta}$ of them. Let X denote the random variable pertaining to the time (in hours) until the first earthquake. Then X is geometric with parameter $p = \lambda\Delta$. Thus

$$P(\text{no earthquake in 10 days}) = P\left(X > \frac{10}{\Delta}\right) = (1 - \lambda\Delta)^{\frac{10}{\Delta}}.$$

   (b) **[10 points]** Let Y denote the number of earthquakes in 10 days. Calculate the expectation of Y and determine the variance of Y. **Solution:** Y is a binomial random variable where $p = \lambda\Delta$ and $n = \frac{10}{\Delta}$. As such,

$$E[Y] = np = 10\lambda.$$

   As for the variance, first note that $np = 10$ and $p << 1$, so we can approximate Y by $\tilde{Y}$, a Poisson random variable of parameter 10λ. With this approximation,

$$\text{var}(Y) \simeq \text{var}(\tilde{Y}) = 10\lambda.$$

   (c) **[10 points]** Suppose now that a new town is discovered on another planet, and assume that the statistics governing earthquake occurrences is the same as in the problem description above, except λ is unknown. If 2 earthquakes occur in 3650 days, then determine the maximum likelihood estimate of λ.

   Solution: Let Z denote the random variable pertaining to the number of occurrences in 10 years, which is equivalent to 3650 days. Because $\lambda << 1$, we assume this to be a Poisson random variable of parameter 3650λ. Thus, $P_Y(k) = \frac{(3650\lambda)^k e^{-3650k}}{k!}$ and so for any fixed k, we define the likelihood of λ,

$$\begin{aligned} L(\lambda) &= \frac{1}{k!}(3650\lambda)^k e^{-3650k} = \frac{1}{k!}\exp\left(k\log(3650\lambda) - 3650\lambda\right). \\ \Rightarrow 0 &= L'(\lambda) = \frac{k}{3650\lambda} \times 3650 - 3650 \\ \Rightarrow \hat{\lambda}_{ML} &= \frac{k}{3650} = \frac{2}{3650} \end{aligned}$$