University of Illinois **Spring 2009**

ECE 313: Problem Set 11
Joint Distributions of Random Variables

Due:	Wednesday April 22 at 4 p.m.
Reading:	*Ross* Chapter 6 Sections 1-4; Powerpoint Lecture Slides, Sets 30-35
Noncredit Exercises:	**Chapter 6:** Problems 1-3, 9, 10, 13, 15, 19-23, 40-42; Theoretical Exercises 4, 6; Self-Test Problems 3, 5, 6, 7
Reminders:	Hour Exam II on Monday April 13, 7:00 p.m. – 8:00 p.m. Room 100, Noyes Laboratory
	No class on Friday April 10; time given off in lieu of evening hour exam. However, the TAs will present review sessions for the exam in each of the three classrooms on Friday. There will be no review session on Sunday.

1. **[Low-pass filtering a signal]**
   A signal $x(t) = \exp(-\pi t^2), -\infty < t < \infty$, is the input to an ideal low-pass filter whose transfer function is $H(f) = \text{rect}(f/2)$. Let $y(t)$ denote the output of the filter. Find the *numerical* value of $y(0)$. [Hint: $X(f) = \exp(-\pi f^2), -\infty < f < \infty$.]

2. **[Joint pmfs]**
   The joint pmf $p_{\mathbb{X},\mathbb{Y}}(u,v)$ of $\mathbb{X}$ and $\mathbb{Y}$ is shown in the table below.

$u \rightarrow$ $v \downarrow$	0	1	3	5
4	0	1/12	1/6	1/12
3	1/6	1/12	0	1/12
-1	1/12	1/6	1/12	0

   (a) Find the marginal pmfs $p_{\mathbb{X}}(u)$ and $p_{\mathbb{Y}}(v)$.

   (b) Are $\mathbb{X}$ and $\mathbb{Y}$ independent random variables?

   (c) Find $P\{\mathbb{X} \leq \mathbb{Y}\}$ and $P\{\mathbb{X} + \mathbb{Y} \leq 4\}$.

   (d) Find $p_{\mathbb{X}|\mathbb{Y}}(u \mid 3)$, $\mathsf{E}[\mathbb{X} \mid \mathbb{Y} = 3]$ and $\mathsf{var}(\mathbb{X} \mid \mathbb{Y} = 3)$.

3. **[Drill problem on jointly continuous random variables I]**
   Ross, Problem 6.8, page 287 (Problem 8 on page 313 of the 7th editiion)

4. **[Drill problem on jointly continuous random variables II]**
   The jointly continuous random variables $\mathbb{X}$ and $\mathbb{Y}$ have joint pdf

$$f_{\mathbb{X},\mathbb{Y}}(u,v) = \begin{cases} 2\exp(-u-v), & 0 < u < v < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Sketch the u-v plane and indicate on it the region over which $f_{\mathbb{X},\mathbb{Y}}(u,v)$ is nonzero.

   (b) Find the marginal pdfs of $\mathbb{X}$ and $\mathbb{Y}$.

   (c) Are the random variables $\mathbb{X}$ and $\mathbb{Y}$ independent ?

   (d) Find $P\{\mathbb{Y} > 3\mathbb{X}\}$.

   (e) For $\alpha > 0$, find $P\{\mathbb{X} + \mathbb{Y} \leq \alpha\}$.

   (f) Use the result in part (e) to determine the pdf of the random variable $\mathbb{Z} = \mathbb{X} + \mathbb{Y}$.

5. **[Drill problem on jointly continuous random variables III]**
The jointly continuous random variables $\mathbb{X}$ and $\mathbb{Y}$ have joint pdf

$$f_{\mathbb{X},\mathbb{Y}}(u,v) = \begin{cases} 0.5, & 0 \le u < 1, 0 \le v < 1, \text{ and } 0 \le u+v < 1, \\ 1.5, & 0 \le u < 1, 0 \le v < 1, \text{ and } 1 \le u+v < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

   (a) Find the marginal pdf of $\mathbb{X}$.

   (b) Find $P\{\mathbb{X}+\mathbb{Y} \le 3/2\}$ and $P\{\mathbb{X}^2+\mathbb{Y}^2 \ge 1\}$.

6. **[Average of n independent random variables]**
You will have been taught that when making measurements with instruments in the laboratory, it is best to take several readings and average the values rather than making just a single measurement. This problem explores why this is a good idea, and also considers a case when this idea fails to work.

   We model the measurements of a parameter μ as independent continuous random variables $\mathbb{X}_1$, $\mathbb{X}_2$, ..., $\mathbb{X}_n$. From Equation (3.2) in Section 6.3 of Ross, it follows that the pdf of their sum $\mathbb{Z} = \mathbb{X}_1+\mathbb{X}_2+\cdots+\mathbb{X}_n$ is the *convolution of their pdfs,* and from Theorem 7.1 of Chapter 5 it follows that the pdf of the average $\mathbb{W} = \dfrac{\mathbb{X}_1+\mathbb{X}_2+\cdots+\mathbb{X}_n}{n}$ is given by $f_{\mathbb{W}}(\beta) = n \cdot f_{\mathbb{Z}}(n\beta)$.

   (a) As you learned in ECE 210, convolutions can be computed via Fourier transforms. Use the Fourier transform tables in your ECE 210 text (or other book in the library) to find the Fourier transform of a zero-mean Gaussian pdf $\frac{1}{\sigma\sqrt{2\pi}}\exp(-t^2/2\sigma^2)$.

   (b) Use the time shift theorem of Fourier transform theory to find the Fourier transform of the pdf $\frac{1}{\sigma\sqrt{2\pi}}\exp(-(t-\mu)^2/2\sigma^2)$ from your answer to part (a).

   (c) If each $\mathbb{X}_i$ is a Gaussian random variable with mean μ and variance σ^2, what is the Fourier transform of the pdf of $\mathbb{Z} = \mathbb{X}_1+\mathbb{X}_2+\cdots+\mathbb{X}_n$?

   (d) Use the result of part (c) to show that $\mathbb{Z}$ is a Gaussian random variable with mean $n\mu$ and variance $n\sigma^2$. Note that this is a special case of Proposition 3.2 in Chapter 6 of the textbook. In fact, the analysis that you have done is easily modified to prove Proposition 3.2 very straightforwardly in comparison to the proof in the textbook.

   (e) Use the result of part (d) to deduce that $\mathbb{W} = \mathbb{Z}/n$ is a Gaussian random variable with mean μ and variance σ^2/n. Since most of the probability mass of a Gaussian random variable lies within ± 3 standard deviations of the mean, the value of $\mathbb{W}$, the average of n measurements, is very much more likely to be very close to μ than the value of any individual measurement $\mathbb{X}_i$.

   Now suppose that the $\mathbb{X}_i$ are Cauchy random variables with identical pdfs of the form $\dfrac{1}{\pi(1+(t-\mu)^2)}$, $-\infty < t < \infty$ which have a peak at μ. We now essentially repeat the previous analysis.

   (f) What is the Fourier transform of $\frac{1}{\pi(1+t^2)}$?

   (g) Use the time shift property to deduce the Fourier transform of $\frac{1}{\pi(1+(t-\mu)^2)}$

   (h) Find the Fourier transform of the pdf of $\mathbb{Z} = \mathbb{X}_1+\mathbb{X}_2+\cdots+\mathbb{X}_n$.

   (i) Show that $\mathbb{W} = \dfrac{\mathbb{X}_1+\mathbb{X}_2+\cdots+\mathbb{X}_n}{n}$ has the same Cauchy pdf as the $\mathbb{X}_i$. Thus, if your measurements are centered about μ but have a Cauchy distribution, then the average of n measurements has the *same* pdf as each individual measurement, and is thus no more likely to be close to the mean than the individual measurement.

   Note that Fourier transforms are found in probability theory under the name *characteristic functions* but are not discussed in the textbook.

7. **[Dividing a Poisson stream into two streams]**
Packets arriving at a router are addressed either to Server A or to Server B. Each packet address (A or B) may be regarded as an independent trial of an experiment whose outcomes are A and B with probabilities p and $q = 1 - p$ respectively. The router sends each packet to the appropriate server.

We model the packet arrivals as a Poisson process with arrival rate λ. Let $\mathbb{X} = N(0, T]$ denote the number of packets arriving at the router during $(0, T]$ and $\mathbb{Y} = N_A(0, T]$ the number of packets that are routed to Server A. Then, all the above implies that *given* that $\mathbb{X} = n$, the *conditional* pmf $p_{\mathbb{Y}|\mathbb{X}}(m \mid n)$ of $\mathbb{Y}$ is binomial with parameters (n, p). Of course, $\mathbb{X}$ is a Poisson random variable with parameter λT.

(a) Use the law of total probability to prove that the *unconditional* pmf of $\mathbb{Y}$ is Poisson with parameter $(\lambda p)T$.

(b) What is the conditional pmf $p_{\mathbb{X}|\mathbb{Y}}(n|m)$ of $\mathbb{X}$ given that $\mathbb{Y}$ was observed to have value m?

(c) What is $\mathsf{E}[\mathbb{X} \mid \mathbb{Y} = m]$, the conditional expectation of $\mathbb{X}$ given that $\mathbb{Y}$ was observed to have value m?

8. **[A piece of cake? or a sheet cake with a piece missing?]**
The jointly continuous random variables $\mathbb{X}$ and $\mathbb{Y}$ have pdf

$$f_{\mathbb{X},\mathbb{Y}}(u, v) = \begin{cases} c, & 0 < u < 1,\ 0 < v < 1,\ \max\{u, v\} > \frac{1}{2}, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Sketch the u-v plane and indicate on it the region where $f_{\mathbb{X},\mathbb{Y}}(u, v)$ is nonzero.

(b) Find the value of c.

(c) Find the marginal pdf $f_{\mathbb{Y}}(v)$ of $\mathbb{Y}$.

(d) From your answer to part (c), find $\mathsf{E}[\mathbb{Y}]$.

(e) What is $f_{\mathbb{Y}|\mathbb{X}}(v \mid \alpha)$, the conditional pdf of $\mathbb{Y}$ given that $\mathbb{X} = \alpha$ where $0 < \alpha < \frac{1}{2}$?

(f) What is $f_{\mathbb{Y}|\mathbb{X}}(v \mid \beta)$, the conditional pdf of $\mathbb{Y}$ given that $\mathbb{X} = \beta$ where $\frac{1}{2} < \beta < 1$?

(g) What is $P\{\mathbb{Y} \le \alpha\mathbb{X}\}$ where $0 < \alpha \le 1$?

(h) What is $P\{\mathbb{Y} \le \alpha\mathbb{X}\}$ where $1 < \alpha < \infty$?

(i) If $\mathbb{Z} = \mathbb{Y}/\mathbb{X}$, find $P\{\mathbb{Z} \le \alpha\}$ for all $\alpha,\ 0 < \alpha < \infty$.

(j) Find the pdf $f_{\mathbb{Z}}(\alpha)$. Be sure to specify the value of $f_{\mathbb{Z}}(\alpha)$ for all $\alpha,\ -\infty < \alpha < \infty$.