University of Illinois **Spring 2009**

ECE 313: Problem Set 10
Functions of a random variable, hazard rates, decision making

Due: Wednesday April 8 at 4 p.m.
Reading: *Ross* Sections 5.5 – 5.7

1. **[System reliability]**
The lifetime of a system with hazard rate $\lambda(t) = bt$ is a Rayleigh random variable X with pdf $f(u) = (bu)e^{-\frac{bu^2}{2}}$ for $u > 0$.

   (a) What is the *median* lifetime of the system? (Recall that the median is the value u s.t. $F_X(u) = 1/2$.) Is the median larger or smaller than the mean lifetime? How do these compare to the *mode* of the lifetime?

   (b) It is observed that the system fails at time $X = t$. What is the maximum-likelihood estimate of the parameter b? (Note: the maximum-likelihood estimate of b is the value of b that maximizes the likelihood $f(t)$.)

2. **[Functions of a random variable I]**
Let the random variable X be uniform on the unit interval $(0, 1)$. Find the pdf/pmf of $Y = -\ln X$.

3. **[Functions of a random variable II]**
Let X be a random variable with pdf given by $f_X(u) = \frac{1}{2}u^{-2}$ for $|u| \geq 1$, and zero otherwise.

   (a) Determine the pdf/pmf of Y when $Y = \text{sign}(X + 2)$ with

$$\text{sign}(x) = \begin{cases} 1 & : \quad x \geq 0 \\ -1 & : \quad x < 0 \end{cases}.$$

   (b) Find the pmf/pdf for Z when Z is defined as

$$Z = \begin{cases} 2|X| & : \quad |X| \leq 2 \\ X^2 & : \quad |X| > 2 \end{cases}.$$

4. **[Random spheres]**
Consider a sphere whose radius is a random variable, R, with pdf $f_R(u) = 2u$ for $0 < u < 1$.

   (a) What is the average radius of the sphere?

   (b) What is the average volume of the sphere? If we call a sphere with average radius an average sphere, then does the average sphere have the average volume?

   (c) What is the average surface area of the sphere? Does the average sphere have the average surface area ?

   (d) Show that $E[R] > E[R^2] > E[R^3]$ if the pdf for R is any valid pdf that is nonzero only on the unit interval $(0, 1)$.

5. **[Hypothesis testing]**
The random variable X models a physical parameter. If hypothesis H_0 is true, then $f_0(u)$, the pdf of X, is Gaussian with mean 0 and variance a^2. If hypothesis H_1 is true, then $f_1(u)$, the pdf of X, is Gaussian with mean 0 and variance $b^2 > a^2$.

   (a) The pdf $f_i(u)$ can be thought of as the conditional pdf of X given that H_i has occurred, i.e., $f_i(u) = f_{X|H_i}(u|H_i)$. Suppose that $\pi_0 = \pi_1$. Write an expression for the unconditional pdf of X. Is the unconditional pdf of X a Gaussian pdf?

   (b) What is the likelihood ratio? Simplify your answer.

   (c) What is the maximum-likelihood decision rule? What are the false alarm and missed detection probabilities for this decision rule?

6. **[Target recognition: Gaussian noise]**
Consider an infrared image of a tank in a wooded area. It is known that tanks are warmer than the typical flora in the woods (this particular woods is devoid of all fauna). A pixel in this image can be modeled by a Gaussian random variable, $X \sim \mathcal{N}(\mu_i, \sigma_i^2)$, where the mean and variance depend on whether or not the pixel is a part of a tank. NSA scientists have determined that if the pixel is a part of the tank, then $X \sim \mathcal{N}(100, \sigma_1^2)$, while if the pixel is not a part of the tank $X \sim \mathcal{N}(25, \sigma_0^2)$. Suppose that a particular pixel in the image has value $X = 75$.

   (a) If $\sigma_1^2 = \sigma_0^2$, what is the maximum likelihood decision as to whether or not the pixel is a part of a tank?

   (b) If $\sigma_1^2 = 40$ and $\sigma_0^2 = 20$, what is the maximum likelihood decision as to whether or not the pixel is a part of a tank?

   (c) If $\sigma_1^2 = 10$ and $\sigma_0^2 = 50$ what is the maximum likelihood decision as to whether or not the pixel is a part of a tank?

7. **[Target detection using the Laplacian pdf]**
Let H_0 be the hypothesis that no target is present, and H_1 be the hypothesis that a target is present. If H_0 is true, the pdf of X is $f_0(u) = \frac{1}{2}e^{-|u+1|}, -\infty < u < \infty$ and if H_1 is true, the pdf of X is $f_1(u) = \frac{1}{2}e^{-|u-1|}, -\infty < u < \infty$. Such pdfs are called Laplacian or double exponential pdfs.

   (a) Sketch the two pdfs (be careful... the absolute value signs can be quite tricky).

   (b) Determine the maximum-likelihood decision rule as a threshold test on the observed value u of the random variable X.

   (c) What are the probabilities of false alarm and missed detection for the maximum-likelihood decision rule of part (b).

   (d) Compute the values of the likelihood ratio (recall, $\Lambda(u) = \frac{f_1(u)}{f_0(u)}$) for $u = -4, -3, -2 \cdots 2, 3, 4$.

   (e) The Bayesian decision rule compares $\Lambda(u)$ to the ratio $\frac{\pi_0}{\pi_1}$. Sow that this decision rule also can be stated in terms of a threshold test on u.

   (f) If $\pi_0 = 2\pi_1$, what is the average probability of error of the Bayesian decision rule?

   (g) What is the average error probability of a decision rule that always decides H_0 regardless of the observed value of X?

   (h) Show that if $\pi_0 > e^2/(e^2 + 1)$ the Bayesian decision rule always decides H_0 regardless of the observed value of X.