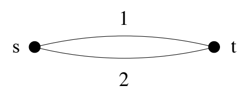
University of Illinois **Spring 2009**

ECE 313: Problem Set 9
Exponential distribution, Poisson random processes, Gaussian random variables

Due:	Wednesday April 1 at 4 p.m.
Reading:	*Ross* Sections 5.1-5.5, and 9.1.
Noncredit Exercises:	*Ross*, Chapter 5, problems 15-34, theoretical exercises 7-14, self-test 5.14, and *Ross*, Chapter 9, theoretical exercises 1-4 and self-test 1&2.

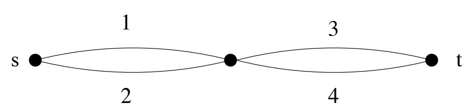
1. [**Lifetime of a network with two parallel links**]
Consider the network with a source node s, terminal node t, and two links as shown.



Suppose that for each i, link i fails at time T_i, where the random variables T_i for different links are independent, exponentially distributed with parameter $\lambda = 1$. The network fails at time T, which is the first time that at least one link has failed on each path from s to t.

   (a) Find $P\{T_1 > 1\}$ and $P(T_1 > 2|T_1 > 1)$. Which is larger, or are they equal?
   (b) Find $P\{T > t\}$ for $t \geq 0$. (Note that $F_T(t) = 1 - P\{T > t\}$.)
   (c) Find $P\{T > 1\}$ and $P(T > 2|T > 1)$. Which is larger?
   (d) Find the pdf of T.
   (e) Find the mean of T. (Hint: Using either the pmf, f_T, or the area formula for $E[X]$ based on F_T, should work well.)

2. [**Lifetime of a network with four links**]
Consider the network with a source node s, terminal node t, and four links as shown.



Suppose that for each i, link i fails at time T_i, where the random variables T_i for different links are independent, exponentially distributed with parameter $\lambda = 1$. The network fails at time T, which is the first time that at least one link has failed on each path from s to t.

   (a) Find $P\{T > t\}$ for $t \geq 0$. (Note that $F_T(t) = 1 - P\{T > t\}$.)
   (b) Find the pdf of T.
   (c) Find the mean of T. (Hint: Using either the pmf, f_T, or the area formula for $E[X]$ based on F_T, should work well.)

3. [**Arrival times for a Poisson process**]
Calls arrive to a cell in a certain wireless communication system according to a Poisson process with arrival rate $\lambda = 2$ calls per minute. Measure time in minutes and consider an interval of time beginning at time $t = 0$. Let $N(t)$ denote the number of calls that arrive up until time t.

(a) What is the pmf of $N(t)$ for a fixed $t > 0$?

(b) Find the probabilities of each of the events:
E_1="No calls arrive in the first 3.5 minutes"
E_2="The first call arrives after time $t = 3.5$"
E_3="two or fewer calls arrive in the first 3.5 minutes"
E_4="the third call arrives after time $t = 3.5$"
E_5="the third call arrives after time t" (for general $t > 0$) (Use only finitely many terms.)
E_6="the third call arrives before time t" (for general $t > 0$) (Hint: $P(E_6) = 1 - P(E_5)$.)

(c) Note that, as a function of t, $P(E_6)$ is the CDF for the time of the third call arrival. Differentiate this function with respect to t to find the pdf for the time of the third arrival. Simplify your answer. You should get the gamma density with parameters $(3,\lambda)$.

(d) What is the expected arrival time of the tenth call?

4. **[Poisson process probabilities]**
Consider a Poisson process with rate $\lambda > 0$.

(a) Find the probability there is exactly one arrival in each of the intervals (0,1], (1,2], and (2,3].

(b) Find the probability that there are two arrivals in the interval $(0, 2]$ and two arrivals in the interval $(1, 3]$. (Hint: The event in (a) implies the event in (b). What other ways can the event in (b) be true?)

(c) Find the probability that there are two arrivals in the interval (1,2], given that there are two arrivals in the interval (0,2] and two arrivals in the the interval (1,3].

5. **[Calculating probabilities of events involving a Gaussian random variable]**
Let X be a $N(10, 16)$ random variable (i.e. a Gaussian random variable with mean 10 and variance 16). Find the numerical values of the following probabilities.

(a) $P(X \geq 15)$

(b) $P(X \leq 5)$

(c) $P(X^2 \geq 400)$

(d) $P(X = 2)$.

6. **[Normal approximation to central term of binomial distribution]**
Let n be a positive even integer, and let X be a binomial random variable with parameters n and $p = 0.5$. This problem focuses on $P\{X = \frac{n}{2}\}$. The continuity correction for approximating the binomial distribution by the normal distribution begins by writing this same probability as $P\{\frac{n-1}{2} \leq X \leq \frac{n+1}{2}\}$.

(a) Find the normal approximation to $P\{X = \frac{n}{2}\}$, using the continuity correction. Your anwer should involve n and the standard normal CDF, Φ.

(b) Find the constant c such that $\sqrt{n}P\{X = \frac{n}{2}\} \to c$ as $n \to \infty$, assuming you can replace $P\{X = \frac{n}{2}\}$ by its normal approximation found in part (a). This suggests that $P\{X = \frac{n}{2}\} \approx \frac{c}{\sqrt{n}}$ for large n. (Hint: If a function f is differentiable at a point u, then $\frac{f(u+h)-f(u-h)}{2h} \to f'(u)$ as $h \to 0$.)

(c) Compute the true value of $P\{X = \frac{n}{2}\}$, the approximation found in part (a), and the approximation found in part (b), for $n = 30$.