ECE 313: Problem Set 8
Cumulative Distribution Functions, Continuous Random Variables

Due:	Wednesday March 18 at 4 p.m.
Reading:	*Ross* Chapter 4.10, Chapter 5; Powerpoint Lecture Slides, Sets 21-24
Noncredit Exercises:	**Chapter 5:** Problems 1-3, 5, 6, 15-19, 23-25, 32-34; Theoretical Exercises 1, 8; Self-Test Problems 1-4
Reminders:	No class on Friday March 13 on account of EOH No class on Friday March 20 (time off for evening exams)

1. **[Maximization of the newsboy's profit: Generalization of Problem 4 of Problem Set 4]**
Each day, a newsboy buys newspapers from the publisher for c_1 cents each, sells them for c_2 cents each, and recycles the unsold papers (if any) getting c_3 cents for each. Note that $c_2 > c_1 > c_3$. Let H denote the number of papers that the newsboy purchases each day. The demand for papers is a discrete random variable $\mathbb{X}$ that takes on nonnegative integer values. Do *NOT* assume that $\mathbb{X}$ is a binomial random variable as in Problem 4 of Problem Set 4. Let $F(u)$ denote the CDF of $\mathbb{X}$.

   (a) Express the probability that the newsboy is able to sell all H papers in terms of $F(u)$.

   (b) One day, the newsboy decides to buy *one* additional paper in the hopes of selling it and increasing his profit. Express the probability that he is unable to sell the additional paper in terms of $F(u)$. Be sure you understand the difference between "not being able to sell the $(H + 1)$-th paper" and "being able to sell all H papers but not the extra $(H + 1)$-th paper."

   (c) Find $A(H+1)$, the average *additional* profit from the sale of the extra (that is, $(H+1)$-th) paper.

   (d) Use the properties of CDFs to show that $A(H + 1)$ is a non-increasing function of H and that $A(H + 1) < 0$ for sufficiently large values of H.

   (e) What choice of H *maximizes* the newsboy's average daily profit? Call this value of H as H_0.

   (f) *Now* assume that the numerical values of c_1, c_2, c_3 and the pmf of $\mathbb{X}$ are as specified in Problem 4 of Problem Set 4, and find the numerical values of H_0 and the (maximum) average daily profit thus achieved. A spreadsheet or MATLAB calculation should be used in this part.

2. **[Expected values from CDFs]**
Find the expected value $\mathsf{E}[\mathbb{X}]$ of the *mixed* random variable $\mathbb{X}$ of Problem 3 of Problem Set 7. **Hint:** Read the last few slides of Lecture 21 and the first few slides of Lecture 22 before starting this problem.

3. **[Exercises on properties of pdfs]**
Nine functions $f(u)$ are shown below. Note that in each case, $f(u) = 0$ for all u not in the interval specified. In each case,

   - determine whether $f(u)$ is a valid probability density function (pdf).
   - If $f(u)$ is not a valid pdf, determine if there exists a constant C such that $C \cdot f(u)$ is a valid pdf.

   (a) $f(u) = 2u, \quad 0 < u < 1.$ (b) $f(u) = |u|, \quad |u| < \frac{1}{2}$
   (c) $f(u) = 1 - |u|, \quad |u| < 1,$ (d) $f(u) = \ln u, \quad 0 < u < 1.$ Hint: $\ln u$ can be integrated by parts.
   (e) $f(u) = \ln u, \quad 0 < u < 2,$ (f) $f(u) = \frac{2}{3}(u - 1), \quad 0 < u < 3,$
   (g) $f(u) = \exp(-2u), \quad u > 0.$ (h) $f(u) = 4\exp(-2u) - \exp(-u), \quad u > 0,$
   (i) $f(u) = \exp(-|u|), \quad |u| < 1,$

4. **[Lifetime of a device]**
Problem 4 on page 224 of Ross.

5. **[Maximizing profits, Part II]**
The weekly demand (measured in thousands of gallons) for gasoline at a rural gas station is a random variable $\mathbb{X}$ with probability density function

$$f_{\mathbb{X}}(u) = \begin{cases} 5(1-u)^4, & 0 \le u \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Let C (in thousands of gallons) denote the capacity of the tank (which is re-filled weekly.)

   (a) If $C = 0.5$, what is the probability that the weekly demand for gasoline can be satisfied? Note that if your answer is (say) 0.666..., then, in the long run, the gas station can supply the weekly demand two weeks out of three.

   (b) What is the minimum value of C required to ensure that the probability that the demand exceeds the supply is no larger than 10^{-5} ?

   (c) Suppose that the owner makes a gross profit of \$0.64 for each gallon of gasoline sold. Let $\mathbb{Y}$ denote the amount of gasoline sold per week. How is $\mathbb{Y}$ related to $\mathbb{X}$, the weekly demand for gasoline? (Hint: the owner cannot sell more gasoline each week than the tank can hold!) What is the *average* weekly gross profit and how does it behave as a function of C, the tank capacity?

   (d) Suppose that the owner pays \$$20C$ as weekly rent on a tank of capacity $1000C$ gallons. Note that $0 \le C \le 1$. (Why is a tank larger than 1000 gallons not needed?) What is the average weekly *net* profit and what value of C maximizes the average weekly net profit?