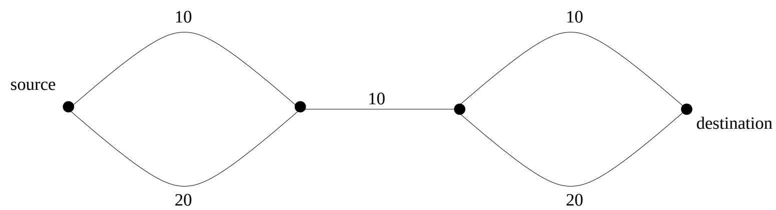
ECE 313: Problem Set 7
Independence, Reliability, and Fault Tolerance

Due: Wednesday March 11 at 4 p.m.
Reading: *Ross* Sections 3.4, 4.10.

1. **[Network capacity and failure]**
Consider the following network, with link capacities (in packets) as shown. Assume that each link fails independently with probability $p = 1/2$.



   (a) What is the probability that a message can be sent successfully from the *source* to the *destination*?

   (b) Let X denote the number of packets that can be sent from the *source* to the *destination*. Determine the pmf for X.

   (c) Determine the expected value of X.

2. **[Practical lessons for gamblers]**
The dice game of *craps* begins with the player (called the *shooter*) rolling two fair dice. If the sum is 2, 3, or 12, the shooter loses the game. If the sum is a 7 or 11, the shooter wins the game. If the sum is any of 4, 5, 6, 8, 9, 10, then the shooter has neither won nor lost (as yet). The number rolled is called the shooter's *point*, and what happens next is described in parts (b) and (c) below.

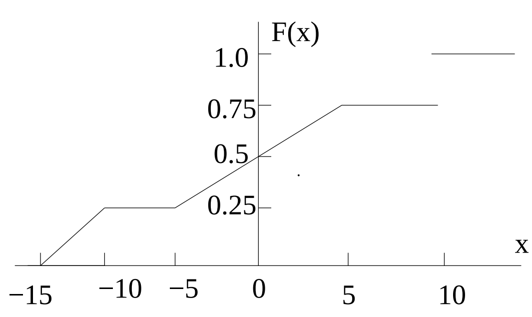
   (a) What is the probability that the shooter wins the game on the first roll? What is the probability that the shooter loses the game on the first roll? What is the probability that the shooter's point is $i, i \in \{4, 5, 6, 8, 9, 10\}$?

   (b) Suppose that the shooter's point is i. The shooter rolls the dice again. If the result is i, the shooter is said to have *made the point* and wins the game. If the result is 7, the shooter loses the game (craps out). If the result is anything else, the shooter rolls the dice again. This continues until the shooter either makes the point or craps out. For each $i \in \{4, 5, 6, 8, 9, 10\}$, compute the probability that the shooter wins the game. Note that these are *conditional* probabilities of winning given that the shooter's point is i.

   (c) Conditioned on the shooter's point being i, what is the expected number of dice rolls till the game ends? (Note: one dice roll = rolling two dice simultaneously). What is the expected number of dice rolls in a game of craps? What is the (unconditional) probability of winning at craps?

   (d) If the shooter's point is 8, then side-bets are offered at 10 to 1 odds that the shooter will make the point *the hard way* by rolling $(4, 4)$. Is this a fair bet? (Remember that 10 to 1 odds means if you bet a dollar, you will either lose the dollar, or you will win ten dollars (and will also get your original dollar back, of course!)).

3. **[Cumulative distribution functions: Part I]**
Let X be a random variable with the cumulative distribution function shown. Compute the following probabilities (a) $P[X \leq 1]$, (b) $P[X > 1]$, (c) $P[|X| \leq 3]$, and (d) $P[X^2 \leq 9]$.



4. **[Voting Blocks]**
The Senate of a certain country has 100 members consisting of 43 Conservative Republicans, 21 Conservative Democrats, 12 Liberal Republicans, and 24 Liberal Democrats. Before each vote, the groups caucus separately. Each group decides independently of the other groups whether to support or oppose the motion. All members of the group then vote in accordance with the caucus decision.

   (a) Let A, B, C, and D respectively denote the events that the four groups vote to eliminate all income taxes on capital gains. Suppose that the probabilities of these independent events are $P(A) = 0.9, P(B) = 0.6, P(C) = 0.5$ and $P(D) = 0.2$. What is the probability that the bill passes?

   (b) The President vetoes the bill as a budget-breaker. Let E, F, G, and H respectively denote the independent events that the four groups support the motion to override the veto. If these events have probabilities $P(E) = 0.99, P(F) = 0.4, P(G) = 0.6$, and $P(H) = 0.1$, what is the probability that the motion to override the veto passes ? NOTE: A simple majority (51 or more votes) is required to pass a bill, and a two-thirds majority (67 or more votes) to override a veto.

5. **[Fault tolerance]**
The ToyAuto Company needs to decide which of the following two methods provides more reliable transportation:

   - a single gigantic car with N engines, N transmissions, N brakes, ... etc. that works (i.e. provides us with transportation) as long as *at least one* of its engines and *at least one* of its transmissions, and *at least one* of its brakes ... works.
   - N separate ordinary cars that fail as soon as any one of their parts fail, but which together provide us with transportation as long as at least one car is in working condition.

   Each car is made of M different types of parts, and (at least) one part of each different type must work for the car to work. Each part fails with probability p and all the failures are independent events.

   (a) For each method, find the probability of system failure in terms of p, N and M

   (b) Suppose that $M = 5$ and $p = 0.2$. If it is desired that the system failure probability be less than 0.001, what should N be with each method?

   (c) Repeat part (b) assuming that $M = 1000$.

6. **[Failure rate function for discrete random variables]**
Suppose an item, such as a light bulb, a sensor, or a satellite, will eventually fail. Suppose the time of failure, in some appropriate units of time, is modeled as a random variable T, with values in the set of positive integers. The *failure rate function* λ_T is the function $\lambda_T = (\lambda_T(k) : k \geq 1)$, defined by $\lambda_T(k) = P(T = k|T \geq k)$ if $P(T \geq k) > 0$. If $P(T \geq k) = 0$, then $\lambda_k(T)$ is not defined.

   (a) Give a general expression for λ_T in terms of the pmf, p_T.

   (b) Note that if $\lambda_T(k)$ is defined for a given value of k, then its value is in the interval $[0, 1]$ (this is true for any conditional probability). What does it mean about the distribution of T if $\lambda_T(k_f) = 1$ for some value $k_f \geq 1$?

   (c) Find λ_T explicitly in case $p_T(k) = \frac{1}{n}$ for $1 \leq k \leq n$. (Hint: $\lambda_T(k)$ is not well defined for $k \geq n+1$ for this example.)

   (d) Find the pmf p_T in case $\lambda_T(k) = p$ for all $k \geq 1$, where p is a constant with $0 < p \leq 1$. (Hint: begin by finding $p_T(1)$ and $p_T(2)$.)

   (e) A network consists of two links in series. Let T be the time the network fails, which is the first time that at least one of the links fails. Assume the two links fail independently, with the failure rate function of link i being $\lambda_i = (\lambda_i(k) : k \geq 1)$, for $i = 0$ or $i = 1$. Express the failure rate function of the network, λ_T, in terms of the functions λ_1 and λ_2. (Hint: Make sure $\lambda_T(k)$ is in the interval $[0, 1]$ if $\lambda_1(k)$ and $\lambda_2(k)$ are both defined. You should also have $\lambda_T(k) = 1$ if $\lambda_i(k) = 1$ for $i = 1$ or $i = 2$.)

7. **[Cumulative distribution functions: Part I]**
Let X be the discrete random variable that denotes the number shown on the top face of a die, and suppose that that $p_X(u_i)$ is propotional to u_i. Sketch the cumulative distribution function for X.