University of Illinois **Spring 2009**

ECE 313: Problem Set 6
Hypothesis testing

Due:	Wednesday, March 4, at 4 p.m.
Reading:	*Notes on Decision Making for ECE 313 Spring 2009,* distributed in class. Supplemental reading, *Chapter 5: Decision Making under Uncertainty,* can be found on the class compass page.
Reminder:	Hour Exam I, Monday March 2, 7 p.m.-8 p.m., 100 Noyes Lab. One 8.5"×11" sheet of notes allowed. The exam will cover the reading assignments, lectures, and problems associated with problem sets 2-5.

1. [**Ternary channel**]
A communication system transmits one of three signals, s_1, s_2, or s_3. Due to channel noise, the received signal can be different from the transmitted signal, with probabilities given by the likelihood matrix:

		Receive, j		
		s_1	s_2	s_3
	s_1	0.8	0.1	0.1
Send, i	s_2	0.05	0.9	0.05
	s_3	0.02	0.08	0.9

For example, $P(s_3 \text{ received} \mid s_1 \text{ sent}) = 0.1$. A decision rule specifies, for each possible received signal, a corresponding choice of transmitted signal.

   (a) Indicate the ML decision rule.

   (b) Suppose that s_i is transmitted with prior probability π_i, for $1 \leq i \leq 3$, such that $(\pi_0, \pi_1, \pi_2) = (0.4, 0.2, 0.4)$. Find the joint probability matrix, indicate the MAP decision rule, and find the average probability of error, p_e, for the MAP rule.

2. [**Detection problem with Poisson distributed observations**]
A certain deep space transmitter uses on-off modulation of a laser to send a bit. The result is that: if a zero is sent, the number of photons, X, arriving at the receiver has the Poisson distribution with mean $\lambda_0 = 2$; and if a one is sent, X has the Poisson distribution with mean $\lambda_1 = 6$.

   (a) Describe the ML decision rule. Express it as directly in terms of X as possible.

   (b) Describe the MAP decision rule under the assumption that sending a zero is a priori five times more likely than sending a one (i.e. $\pi_0/\pi_1 = 5$). Express the rule as directly in terms of X as possible.

3. [**Detection problem with geometric distribution vs. Poisson distribution**]
Consider a detection problem with the following two hypotheses for an observation X :

   $H_0 : X$ has the Poisson distribution with parameter $\lambda = 10$.

   $H_1 : X$ has the geometric distribution with parameter $p = 0.1$.

(a) To get some intuition about this problem, calculate the mean, variance, and standard deviation of X under H_0 and under H_1.

(b) Describe the ML decision rule. Express it as directly in terms of X as possible.

(c) Describe the MAP decision rule, under the assumption that H_0 is a priori five times more likely than H_1 (i.e. $\pi_0/\pi_1 = 5$). Express the rule as directly in terms of X as possible.

4. **[Sensor fusion]**
Two motion detectors are used to detect the presence of a person in a room, as part of an energy saving temperature control system. The first sensor outputs a value X and the second sensor outputs a value Y. Both outputs have possible values $\{0, 1, 2\}$, with larger numbers tending to indicate that a person is present. Let H_0 be the hypothesis a person is absent and H_1 be the hypothesis a person is present. The likelihood matrices for X and for Y are shown:

	$X = 0$	$X = 1$	$X = 2$
H_1	0.1	0.3	0.6
H_0	0.8	0.1	0.1

	$Y = 0$	$Y = 1$	$Y = 2$
H_1	0.1	0.1	0.8
H_0	0.7	0.2	0.1

For example, $P[Y = 2|H_1] = 0.8$. Suppose that, given one of the hypotheses is true, the sensors provide conditionally independent readings, so that
$P[X = i, Y = j|H_k] = P[X = i|H_k]P[Y = j|H_k]$ for $i, j \in \{0, 1, 2\}$ and $k \in \{0, 1\}$.

(a) Find the likelihood matrix for the observation (X, Y) and indicate the ML decision rule. To be definite, break ties in favor of H_1. (Note: The matrix specifies $P[X = i, Y = j|H_k]$ for each hypothesis H_k and for each possible observation value (i, j). The matrix has two rows, the first row corresponding to H_1 and the second row corresponding to H_0. The matrix has nine columns of numbers, each corresponding to one of the nine possible values of the observation (X, Y): $(0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), (2, 2)$. The sum of the entries in each row is one. The ML rule can be indicated by underlining the larger number in each column.)

(b) Find $p_{\text{false\_alarm}}$ and p_{miss} for the ML rule found in part (a).

(c) Suppose, based on past experience, prior probabilities $\pi_1 = P[H_1] = 0.2$ and $\pi_0 = P[H_0] = 0.8$ are assigned. Compute the joint probability matrix and indicate the MAP decision rule. (The matrix specifies $P[X = i, Y = j, H_k]$ for each hypothesis H_k and for each possible observation value (i, j). The 18 numbers in the matrix sum to one. The MAP decision rule can be indicated by underlining the larger number in each column.)

(d) For the MAP decision rule, compute $p_{\text{false\_alarm}}$, p_{miss}, and the unconditional probability of error $p_e = \pi_0 p_{\text{false\_alarm}} + \pi_1 p_{\text{miss}}$. (Hint: The conditional probability p_{miss} for the MAP decision rule is the sum of all the entries in the H_1 row of the *likelihood matrix* that correspond to entries *not* underlined in the H_1 row of the *joint probability matrix.* The conditional probability $p_{\text{false\_alarm}}$ is computed similarly, using the H_0 rows.)

(e) Using the same priors as in part (c), compute the unconditional error probability, p_e, for the ML rule from part (a). Is it smaller or larger than p_e found for the MAP rule in (d)?