University of Illinois **Spring 2009**

ECE 313: Problem Set 5
Conditional Probability, Law of Total Probability, and Bayes' Formula

Due:	Wednesday February 25 at 4 p.m.
Reading:	*Ross* Chapter 3; Powerpoint Lecture Slides, Sets 9-14
Noncredit Exercises:	**Chapter 3:** Problems 1, 2, 5, 10, 12, 16, 31, 38, 39, 51, 52 Theoretical Exercises 1, 2, 8, 16; Self-Test Problems 1-14.
Reminder:	Hour Exam I on Monday March 2, 7:00 p.m. – 8:00 p.m. Room 100, Noyes Laboratory

1. **[Picking a random subset]**
   The experiment consists of drawing a *subset* of size k at random from a set of n distinct objects.

   (a) How many different subsets of size k are there of the set of n objects? What is the probability of drawing a specific subset, say $\{\omega_3, \omega_7, \omega_{23} \ldots\}$ of size k?

   (b) Now suppose that we draw objects one at a time *without replacement* until k objects have been drawn. Let A_1, A_2, $\ldots$, A_k denote the events that the first, second, $\ldots$ k-th draw respectively gives us *some* element of the specific subset of interest. Note that we get our desired subset if and only if the event $A_1 A_2 \cdots A_k$ occurs. What is $P(A_1)$? What is $P(A_2 \mid A_1)$? What is $P(A_3 \mid A_1 A_2)$? More generally, what is $P(A_i \mid A_1 A_2 \cdots A_{i-1})$? Use these results to calculate $P(A_1 A_2 \cdots A_k)$ and compare to the result of part (a).

2. **[The beginning of spring training]**
   A baseball pitcher's repertoire is limited to *fastballs* (event F), *curve balls* (event C) and *sliders* (event S). It is known that $P(C) = 2P(F)$. Let H denote the event that the batter hits the ball. Whether H occurs or not depends on the pitch, and it is known that $P(H \mid F) = 2/5, P(H \mid C) = 1/4$, and $P(H \mid S) = 1/6$. If $P(H) = 1/4$, what is $P(C)$?

3. **[Exercises in Conditional Probability]**
   This problem on conditional probability has four unrelated parts:

   (a) The law of total probability states that $P(A) = P(A \mid C)P(C) + P(A \mid C^c)P(C^c)$. Show that this result still holds when *everything* is conditioned on event B, that is, prove that

   $$P(A \mid B) = P(A \mid BC)P(C \mid B) + P(A \mid BC^c)P(C^c \mid B).$$

   (b) If $P(A \mid B) = 0.3, P(A^c \mid B^c) = 0.4$, and $P(B) = 0.7$, find $P(A \mid B^c), P(A)$, and $P(B \mid A)$.

   (c) If $P(E) = \frac{1}{4}, P(F \mid E) = \frac{1}{2}$, and $P(E \mid F) = \frac{1}{3}$, find $P(F)$.

   (d) If $P(G) = P(H) = \frac{2}{3}$, show that $P(G \mid H) \geq \frac{1}{2}$.

4. **[Polya's urn model: a classic result in probability theory]**
   An urn contains r red and g green balls. Two balls are drawn at random from the urn, with the first ball being returned to the urn (which is then shaken well, not stirred) before the second ball is drawn. Let R_1 and R_2 respectively denote the events that the first and second balls are red.

   (a) What are the values of $P(R_1)$ and $P(R_2)$?

   (b) Now suppose that when the first ball is returned to the urn, c *additional* balls of the *same color* are also put into the urn (which is then shaken well before the second ball is drawn.) Clearly $P(R_1)$ is the same as before, but what is $P(R_2)$ now? Remember that the urn now contains $r + g + c$ balls. Simplify your answer and compare to the value of $P(R_2)$ that you obtained in part (a).

(c) For the experiment of part (b), what is the conditional probability that the urn contained $r + c$ red balls given that R_2 occurred?

5. **[Fred and Wilma learn the Law of Total Probability and escape from Guantanamo Bay]**
In this problem, you will use the law of total probability to re-analyze the first game played by Fred and Wilma on Problem 6 of Problem Set 2. Let F and W respectively denote the events that Fred and Wilma win a game, and let H denote the event that Fred tosses a Head on his first attempt.

Consider the game in parts (a)-(c) of Problem 2.6 in which the first one to toss a Head wins the game.

(a) An easy one: What are the values of $P(F \mid H)$ and $P(W \mid H)$?

(b) If the event $H^c = T$ occurs, Fred passes the coin to Wilma to toss for her very first time. Explain why at this point in the game, the probability of Wilma winning the game is exactly $P(F)$, that is, explain why $P(W \mid H^c) = P(W \mid T) = P(F)$. Similarly, explain why $P(F \mid H^c) = P(F \mid T) = P(W)$.

(c) Use the law of total probability to show that $P(F) = p + qP(W)$, $P(W) = qP(F)$, and hence $P(F) = \frac{1}{1+q}$ and $P(W) = \frac{q}{1+q}$ as before.

Wilma now proposes that instead of Fred tossing first in every game, the *loser of a game gets to toss first in the next game.* Fred agrees but insists that he will toss first in the first game. Let F_n and W_n respectively denote the events that Fred and Wilma win the n-th game.

(d) A couple of easy ones: Is it true that Fred and Wilma *always* alternate in tossing the coin regardless of whether a coin toss ends a game or not? What are the values of $P(F_1)$ and $P(W_1)$?

(e) Express $P(F_{n+1})$ and $P(W_{n+1})$ in terms of $P(F_n)$ and $P(W_n)$.

(f) The answers to part (e) are a pair of *difference* equations for $P(F_n)$ and $P(W_n)$. Show that the solutions are $P(F_n) = \frac{1}{2}\left[1 - \left(-\frac{1-q}{1+q}\right)^n\right]$ and $P(W_n) = \frac{1}{2}\left[1 + \left(-\frac{1-q}{1+q}\right)^n\right]$ and that both values converge to $\frac{1}{2}$ as $n \to \infty$ so that if Fred and Wilma are still playing, the game is quite close to being fair by now!

[Hint: If you never learned how to solve difference equations in Math 286 or ECE 410, assume that for *each integer* $n \geq 1$, it is possible to express $P(F_n)$ as $a + b\alpha^n$. Substitute into the difference equation and use the fact that equality must hold for *all* n to find a and α. The value of b is obtained from the initial condition $P(F_1) = \frac{1}{1+q}$. The solution for $P(W_n)$ is similar but the initial condition is $P(W_1) = \frac{q}{1+q}$.]

6. **[The Sun Also Rises, But Maybe The Cake Doesn't]**
Problem 42, Chapter 3 of Ross.

7. **[The (in)famous Monty Hall problem]**
Monty Hall, the host of the TV game show "Let's Make A Deal" shows you three curtains. One curtain conceals a car, while the other two conceal goats. All three curtains are equally likely to conceal the car. He offers you the following "deal": pick a curtain, and you can have whatever is behind it. When you pick a curtain, instead of giving you your just deserts, Monty (who knows where the car is) opens one of the remaining curtains to show you that there is a goat behind it, and offers the following "new, improved deal": you can either stick with your original choice, or switch to the remaining (unopened) curtain. Amidst the deafening roars of "Stand pat" and "Switch, you idiot" from the crowd, Monty points out that previously your chances of winning were 1/3. Now, since you know that the car is behind one of the two unopened curtains, your chances of winning have increased to 1/2, and thus the new improved deal is indeed better.

(a) Let A denote the event that your first choice of door has the car behind it. What is $P(A)$?

Let B denote the event that your second choice of door has the car behind it.

(b) *Your strategy is to always stay put* and so your second choice is the same as your first choice. What is $P(B \mid A)$ in this case? What is $P(B \mid A^c)$? Use these results to find $P(B)$ for the stay-put strategy.

(c) *Your strategy is to always switch* and so your second choice is the other unopened door. What is $P(B\ midA)$ in this case? What is $P(B \mid A^c)$? Use these results to find $P(B)$ for the always-switch strategy.

(d) *Your strategy is to pick randomly* and so your second choice is equally likely to be either unopened door. What is $P(B \mid A)$ in this case? What is $P(B \mid A^c)$? Use these results to find $P(B)$ for the pick-randomly strategy. Is Monty correct in asserting that if you choose randomly between the two unopened curtains, you have a probability of winning of $1/2$?

(e) Having disposed of your goat, you return the next day to the show, and this time, Monty calls you *and* your friend to come on down and choose one curtain each. Which is better: to be the first to pick a curtain or the second? Or does it not make a difference? This time, Monty opens the curtain chosen by your friend to reveal a goat and sends him back to his seat. He now asks whether you want to stick with your original choice or switch to the the third (unchosen) curtain. Which choice gives you a larger chance of winning the car?

Note: The rules of the game of parts (a)-(d) are that Monty, who knows which curtain conceals the car, always opens one of the two unchosen curtains and he always offers the "new improved deal," that is, he never opens a curtain to reveal the prize (saying "Oops, you lose; return to your seat."). In the game of part (e), he always opens one of the chosen curtains to eliminate one of the contestants and then always offers the other contestant the chance to switch.