University of Illinois **Spring 2009**

ECE 313: Problem Set 3
Functions of discrete random variables, and parameter estimation

Due:	Wednesday February 11 at 4 p.m..
Reading:	*Ross* Sections 4.1-4.6.
Noncredit Exercises:	Chapter 4, *Ross*, Problems 1-42, Theoretical Exercises 1-9, and Self Test problems 1-10.

1. [**Mean and standard deviation of a complicated random variable**]
(Use a spreadsheet, Matlab, calculator, or your favorite computer language for this problem.) Suppose two fair dice are rolled independently, so the sample space is $\Omega = \{(i,j) : 1 \leq i,j \leq 6\}$ and all outcomes are equally likely. Let X be the random variable defined by $X(i,j) = (i-j)^3 + (i-2j)^2$.

   (a) Calculate $E[X]$. (Hint: This is just the average of the 36 values of X.)

   (b) Calculate $E[X^2]$. (Hint: This is just the average of the 36 values of X^2.)

   (c) Using your answers to (a) and (b), calculate the standard deviation of X. (Note: If you were to apply the STDEV function of the Excel spreadsheet to the list of 36 values of X, you would not get the standard deviation of X, because the STDEV function is for *estimating* the standard deviation from n samples, and it uses what is called the $n-1$ rule. The STDEVP function returns the correct value, and can be used for you to check your answer.)

   (d) Find the pmf of X, $p_X(k)$, for $k = 1,2,3$. (Note: In principle, you could compute the complete pmf of X and use it to do (a) and (b) above, but the hints for (a) and (b), based on the law of the unconscious statistician (LOTUS), give a much simpler way to do (a) and (b).)

2. [**Mean and standard deviation of two simple random variables**]
Suppose two fair dice are rolled independently, so the sample space is $\Omega = \{(i,j) : 1 \leq i \leq 6, 1 \leq j \leq 6\}$ and all outcomes are equally likely. Let X be the random variable defined by the outcome of the first roll, $X(i) = i$, and let Y be the random variable defined by $Y(i,j) = \max\{i,j\}$.

   (a) Derive the pmf of X and sketch it.

   (b) Find $E[X]$ and the standard deviation, σ_X, of X. Correct numerical answers are fine, but show your work.

   (c) Derive the pmf of Y and sketch it.

   (d) Find the mean $E[Y]$ and standard deviation, σ_Y, of Y. (Hint: It may be helpful to reuse the spreadsheet or program you used for Problem 1.)

   (e) Which is larger, σ_X or σ_Y? Is that consistent with your sketches of the pmfs?

   (f) Specify a pmf on the set $\{1,2,3,4,5,6\}$ which has a larger standard deviation than X.

3. **[Applying LOTUS for a function of a geometric random variable]**
Suppose X is a geometrically distributed random variable with some parameter p, such that $0 < p \leq 1$. Thus X has the pmf

$$p_X(k) = \begin{cases} p(1-p)^{k-1} & k \geq 1 \\ 0 & \text{else.} \end{cases}$$

   (a) Let $Y = 2^X$. Give an equation that describes the pmf of the random variable Y, and sketch the pmf for $p = 0.5$.

   (b) Express $E[Y]$ as a function of p by using the law of the unconscious statistician (LOTUS). Simplify your answer as much as possible. For what values of p is $E[Y]$ finite?

4. **[The mean and variance of the Poisson distribution vs. binomial distribution]**
Fix $\lambda > 0$. The Poisson pmf with parameter λ is approximately the same as a binomial pmf with parameters n and p, with n much larger than λ and $p = \lambda/n$. Let $n \to \infty$ with $p = \lambda/n$. Show that the mean and variance of the binomial pmf with parameters n and p converge to the mean and variance of a Poission random variable with parameter λ.

5. **[Estimation of the parameter of a Poisson random variable]**

   (a) Suppose X is a Poisson random variable with parameter $\lambda \geq 0$, and that λ is to be estimated by observing the value of X. For example, X could represent the number of photons detected over a fixed interval of time by a deep space photon detector pointing at a faint star. Suppose the value of X observed is $X = n$, for some nonnegative integer n. What is the maximum likelihood estimate, $\widehat{\lambda}$, as a function of n?

   (b) Suppose Y is a Poisson random variable with parameter $\lambda = b^2$, where b is an unknown real-valued parameter, but it is known that $0 \leq b \leq 10$. Suppose the value of Y observed is $Y = m$, for some nonnegative integer m. What is the maximum likelihood estimate, $\widehat{b}$, as a function of m? (Be sure your answer is always within the interval $[0, 10]$.)

6. **[Parameter estimation for a triangular pmf]**
Let n be a positive integer, and suppose X has the pmf

$$p_X(k) = \begin{cases} \frac{2k}{n(n+1)} & 1 \leq k \leq n \\ 0 & \text{else} \end{cases}$$

Suppose n is unknown, but is to be estimated by observation of X. Suppose a particular value k is observed for X. Find the maximum likelihood estimate $\widehat{n}$ of n.