University of Illinois **Spring 2009**

ECE 313: Problem Set 2
Sets, Events, Axioms of Probability and Their Consequences

Due:	Wednesday February 4 at 4 p.m..
Reading:	*Ross* Chapter 1, Sections 1-4; Chapter 2, Sections 1-5
	Powerpoint Lecture Slides, Sets 1-6
Noncredit Exercises:	**Chapter 1:** Problems 1-5, 7, 9;
	Theoretical Exercises 4, 8, 13; Self-Test Problems 1-15.
	Chapter 2: Problems 3, 4, 9, 10, 11-14;
	Theoretical Exercises 1-3, 6, 7, 10, 11, 12, 16, 19, 20; Self-Test Problems 1-8

Yes, the reading and noncredit exercises are the same as in Problem Set 1.

1. [**Two useful MacLaurin series**]

   In Problem 1(d),(e) of Problem Set 1, you showed (we hope!) that $(1+x)^n = \sum_{k=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{k!}x^k$ for positive integer n. Now, according to Eq. 4.2 of *Ross* Chapter 1 (with $y = 1$),

   $$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \quad \text{where} \quad \binom{n}{k} = \frac{n!}{k!(n-k)!}.$$

   (a) Show that your answer to Problem 1 of Problem Set 1 agrees with the result in the textbook.

   (b) Now consider the function $g(x) = (1-x)^{-n}$ where n is a positive integer. Does the MacLaurin series for $g(x)$ contain terms of degree $> n$? If not, what is the highest degree term?

   (c) Use the result of part (b) to write down the MacLaurin series for $(1-x)^{-1}$ and $(1-x)^{-2}$. These results will be used so often in ECE 313 that we recommend that you *memorize them.*

2. [**Subsets of a finite set**]

   Let Ω denote a finite set containing the n elements $\omega_1, \omega_2, \ldots, \omega_n$. The *cardinality* (more informally, the *size*) of a subset $A \subset \Omega$ is the number of elements in A, and is denoted as $|A|$.

   (a) Let $n = 4$. List *all* the subsets of Ω in increasing order of size. How many subsets are there? How many of these subsets are *non-empty* subsets?

   (b) If you listed only 14 or 15 subsets in part (a), please re-do part (a), and this time, include the *empty set* $\emptyset$ and/or Ω as subsets of Ω.

   (c) In your answer to part (a) or (b), verify that for each k, $0 \le k \le 4$, the *total number* of subsets of size k is the same as the *total number* of subsets of size $4-k$. Now explain why for n in general, the total number of subsets of size k is the same as the total number of sets of size $n-k$.

   (d) Each subset A corresponds to a n-bit vector $(x_1, x_2, \ldots, x_n)$ where $x_i = 1$ if $\omega_i \in A$ and $x_i = 0$ if $\omega_i \notin A$. Writing $A \leftrightarrow (x_1, x_2, \ldots, x_n)$ emphasizes the *one-to-one correspondence*: each subset defines a unique n-bit vector, and each n-bit vector defines a unique subset, e.g. with $n = 4$, we have that $\{\omega_1, \omega_3\} \leftrightarrow (1, 0, 1, 0)$.

      i. What n-bit vectors correspond to Ω and to $\emptyset$? What n-bit vector corresponds to A^c?

      ii. If $A \leftrightarrow (x_1, x_2, \ldots, x_n)$, $B \leftrightarrow (y_1, y_2, \ldots, y_n)$, $(A \cup B) \leftrightarrow (z_1, z_2, \ldots, z_n)$ and $(A \cap B) \leftrightarrow (w_1, w_2, \ldots, w_n)$, express the z_i's and w_i's in terms of the x_i's and y_i's. Hint: you may need the logical operators $\vee$ and $\wedge$ that you may have encountered in ECE 290.

      iii. How many different n-bit vectors are there? How many different subsets are there of Ω?

      iv. "They correspond to the nonempty subsets of Ω" Respond as if you are on Jeopardy™: What is the question to which this statement is the answer? How many subsets of Ω are non-empty?

3. **[Unions and Intersections]**
An experiment consists of observing the contents of an 8-bit register. We assume that all 256 byte values are equally likely to be observed.

   (a) Let A denote the event that the least significant bit (LSB) is a ONE. What is $P(A)$?

   (b) Let B denote the event that the register contains 5 ONEs and 3 ZEROes. What is $P(B)$?

   (c) What is $P(A \cup B)$? What is $P(A \cap B)$? What is the probability that exactly one of the two events A and B occurs, i.e. what is $P(A \oplus B)$?

4. **[A problem on sampling without replacement]**
A bag contains n pairs of shoes in distinct styles and sizes. You pick two shoes at random from the bag. Note that this is sampling *without* replacement.

   (a) What is the probability that you get a pair of shoes?

   (b) What is the probability of getting one left shoe and one right shoe?

   Suppose now that $n \geq 2$ and that you choose 3 shoes at random from the bag.

   (c) What is the probability that you have a pair of shoes among the three that you have picked?

   (d) What is the probability that you picked at least one left shoe and at least one right shoe?

5. **[The noisy cereal problem]**
The manufacturer of a cereal tests samples from the production line to see if the samples snap, crackle, and pop as advertised. Let A, B, and C respectively denote the events that the sample being tested *does not* snap, *does not* crackle, and *does not* pop. The manufacturer's tests show that $P(A) = P(C) = 0.3$, $P(B) = 0.2$, $P(AB \cup AC \cup BC) = 0.3$, $P(A \cap B \cap C) = 0.05$, $P(A \cap B) = 0.1$, and $P(A \cap C) = 2P(B \cap C)$.

   (a) Sketch the sample space and indicate on it the events A, B, and C.

   (b) What is the probability that the cereal snaps, crackles, and pops?

   (c) Cereal that fails exactly one test is sold to discount supermarket chains to be marketed under the names Soggies, Blecchies, and Mushies. What is the probability that the sample fails *only* the snap test? *only* the crackle test? *only* the pop test?

6. **[A problem on countably infinite sample spaces]**
A biased coin turns up Heads with probability p where $0 < p < 1$. The probability assigned to a sequence of tosses is simply the product of the probabilities of the individual tosses. For example, $P(HTHH) = p \cdot (1-p) \cdot p \cdot p = p^3(1-p) = P(HHHT) = P(THHH) = P(HHTH)$ and so on.

   Fred and Wilma Flintstone take turns tossing this coin (Fred tosses first) until one of them tosses Heads and wins the game. The sample space of this experiment is thus $\Omega = \{H, TH, TTH, \ldots, T^{n-1}H, \ldots\}$.

   (a) What is $P(T^{n-1}H)$? Verify that $P(\Omega) = 1$.

   (b) Which outcomes in Ω comprise the event "Fred wins game"? which the event "Wilma wins game"?

   (c) Find P(Fred wins game). Now, do a *similar* calculation to find P(Wilma wins game). By *similar* we mean that we want you to sum another series to find P(Wilma wins game). Which is larger: P(Fred wins game) or P(Wilma wins game)? Does the answer depend on the value of p?

   Wilma grows tired of this game (can you tell why?) and proposes a new game in which whoever *matches* the result of the previous toss wins the game. She graciously insists that Fred toss first as before, which he does – poor schmuck – without realizing that he has nothing to match on his first toss! The sample space of this new experiment is thus

$$\Omega = \{HH, TT, THH, HTT, HTHH, THTT, HTHTT, THTHH, \ldots, \}.$$

   (d) Verify that $P(\Omega) = 1$ for this sample space.

   (e) Which outcomes in Ω comprise the event "Fred wins game"? which the event "Wilma wins game"?

   (f) Show that $P(\text{Fred wins game}) < \frac{1}{2}$ for all choices of p, $0 < p < 1$.