University of Illinois **Spring 2009**

ECE 313: Problem Set 6: Solutions
Hypothesis testing

1. [**Ternary channel**]

   (a) To identify the ML rule, simply underline the largest element in each column of the likelihood matrix. In this case, the three entries along the diagonal would be underlined. That is, if s_j is received, the ML rule decides that s_j was also sent, for $j \in \{0, 1, 2\}$.

   (b) The joint probability matrix is obtained by multiplying the i^{th} row of the likelihood matrix by π_i for each i:

		Receive, j		
		s_1	s_2	s_3
Send, i	s_1	$\underline{0.32}$	0.04	0.04
	s_2	0.01	$\underline{0.18}$	0.01
	s_3	0.008	0.032	$\underline{0.36}$

   The MAP rule, obtained by underlining the largest entry of each column of the joint probability matrix, is in this case the same as the ML rule described in part (a). Also, $p_e = 1 - p_c = 1 - (0.32 + 0.18 + 0.36) = 0.14$.

2. [**Detection problem with Poisson distributed observations**]

   (a) The ML rule is to decide a one is sent if $\Lambda(X) \geq 1$, where Λ is the likelihood ratio function, defined by

$$\Lambda(k) = \frac{P(X = k \mid \text{one is sent})}{P(X = k \mid \text{zero is sent})} = \frac{e^{-\lambda_1}\lambda_1^k/k!}{e^{-\lambda_0}\lambda_0^k/k!} = \left(\frac{\lambda_1}{\lambda_0}\right)^k e^{-(\lambda_1-\lambda_0)} = 3^k e^{-4} \approx \frac{3^k}{54.6}$$

   Therefore, the ML decision rule is to decide a one is sent if $X \geq 4$.

   (b) The MAP rule is to decide a one is sent if $\Lambda(X) \geq \frac{\pi_0}{\pi_1}$, where Λ is the likelihood ratio found in part (a). So the MAP rule decides a one is sent if $\frac{3^X}{54.6} \geq 5$, or equivalently, if $X \geq 5$. Note that when $X = 4$, the ML rule decides that a one was transmitted, not a zero, but because zeroes are so much more likely to be transmitted than ones, the MAP rule decides in favor of a zero in this case.

3. [**Detection problem with geometric distribution vs. Poisson distribution**]

   (a)

	mean	variance	standard deviation
H_1	$\frac{1}{p} = 10$	$\frac{1-p}{p^2} = 90$	9.487
H_0	$\lambda = 10$	$\lambda = 10$	3.162

   Note: The means are the same but the distribution under H_1 is more spread out (e.g. its standard deviation is three times larger) than the distribution under H_0. Moreover, on one hand, the geometric pmf with $p = 10$ is slowly decreasing, with values $p(1) = 0.1, p(10) = 0.0387, p(20) = 0.0135$, and $p(30)$=0.00471, for example. On the other hand, the Poisson pmf for $\lambda = 10$ has a peak value at $k = 10$ and falls off rapidly away from $k = 10$.

(b) The ML rule is to decide H_1 is true if $\Lambda(X) \geq 1$, where Λ is given by

$$\Lambda(k) = \begin{cases} \frac{(1-p)^{k-1}p}{e^{-\lambda}\lambda^k/k!} = \frac{pe^\lambda}{1-p}\left(\frac{1-p}{\lambda}\right)^k k! \approx (2447)(0.09)^k k!, & \text{if } k \geq 1, \\ 0, & \text{if } k = 0. \end{cases}$$

Numerical calculation shows that the ML decision rule is to decide H_1 is true if $1 \leq X \leq 5$ or $X \geq 17$.

(c) The MAP rule is to decide H_1 is true if $\Lambda(X) \geq \frac{\pi_0}{\pi_1}$, where Λ is the likelihood ratio function found in part (b). So the MAP rule is to decide H_1 is true if $X = k$ such that $(2447)(0.09)^k(k!) \geq 5$, or equivalently, if $1 \leq X \leq 3$ or $X \geq 20$.

4. **[Sensor fusion]**

(a) The likelihood matrix for observation (X, Y) is the following.

$(X,Y) \rightarrow$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
H_1	0.01	0.01	$\underline{0.08}$	0.03	$\underline{0.03}$	$\underline{0.24}$	0.06	$\underline{0.06}$	$\underline{0.48}$
H_0	$\underline{0.56}$	$\underline{0.16}$	0.08	$\underline{0.07}$	0.02	0.01	$\underline{0.07}$	0.02	0.01

The ML decisions are indicated by the underlined elements. The larger number in each column is underlined, with the tie in case $(0, 2)$ broken in favor of H_1, as specified in the problem statement.

(b) For the ML rule, $p_{\text{false\_alarm}}$ is the sum of the entries in the row for H_0 in the likelihood matrix that are not underlined. So $p_{\text{false\_alarm}} = 0.08 + 0.02 + 0.01 + 0.02 + 0.01 = 0.14$. For the ML rule, p_{miss} is the sum of the entries in the row for H_1 in the likelihood matrix that are not underlined. So $p_{\text{miss}} = 0.01 + 0.01 + 0.03 + 0.06 = 0.11$.

(c) The joint probability matrix is given by

$(X,Y) \rightarrow$	$(0,0)$	$(0,1)$	$(0,2)$	$(1,0)$	$(1,1)$	$(1,2)$	$(2,0)$	$(2,1)$	$(2,2)$
H_1	0.002	0.002	0.016	0.006	0.006	$\underline{0.048}$	0.012	0.012	$\underline{0.096}$
H_0	$\underline{0.448}$	$\underline{0.128}$	$\underline{0.064}$	$\underline{0.056}$	$\underline{0.016}$	0.008	$\underline{0.056}$	$\underline{0.016}$	0.008

The MAP decisions are indicated by the underlined elements in the joint probability matrix. The larger number in each column is underlined.

(d) For the MAP rule,

$$p_{\text{false\_alarm}} = P\left[(X, Y) \in \{(1, 2), (2, 2)\}|H_0\right] = 0.01 + 0.01 = 0.02,$$

and

$$p_{\text{miss}} = P[(X, Y) \notin \{(1, 2), (2, 2)\}|H_1] = 1 - P[(X, Y) \in \{(1, 2), (2, 2)\}|H_1] = 1 - 0.24 - 0.48 = 0.28.$$

Thus, for the MAP rule, $p_e = (0.8)(0.02) + (0.2)(0.28) = 0.072$. (Check that for the MAP rule, p_e is the sum of the probabilities in the joint probability matrix that are not underlined.)

(e) Using the conditional probabilities found in (a) and the given values of π_0 and π_1 yields that for the ML rule: $p_e = (0.8)(0.14) + (0.2)(0.11) = 0.134$, which is larger than the value 0.072 for the MAP rule, as expected from the optimality of the MAP rule for the given priors.