University of Illinois Spring 2009

ECE 313: Final Exam

Friday, May 8, 2009, 8:00 a.m. — 11:00 a.m.
149 National Soybean Research Center, 1101 W. Peabody Drive, Urbana

1. **[20 points]** An urn contains five balls numbered consecutively from 1 to 5. Balls are drawn one at a time (*sampling without replacement*) until the 5 ball is drawn. Let $\mathbb{N}$ denote the number of draws required to draw the 5 ball and let $\mathbb{Z}$ denote the sum of the numbers on the balls drawn prior to the 5 ball.

   (a) **[10 points]** Find the pmf of $\mathbb{N}$.
   Solution: Since sampling without replacement is being used, the 5 ball is equally likely to be drawn on any of the 5 draws, and so $p_{\mathbb{N}}(i) = \frac{1}{5}$ for $i = 1, 2, 3, 4, 5$. Alternatively, $p_{\mathbb{N}}(1) = \frac{1}{5}$, $p_{\mathbb{N}}(2) = \frac{4}{5} \times \frac{1}{4} = \frac{1}{5}$, $p_{\mathbb{N}}(3) = \frac{4}{5} \times \frac{3}{4} \times \frac{1}{3} = \frac{1}{5}$, $p_{\mathbb{N}}(4) = \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} = \frac{1}{5}$, $p_{\mathbb{N}}(5) = \frac{4}{5} \times \frac{3}{4} \times \frac{2}{3} \times \frac{1}{2} \times \frac{1}{1} = \frac{1}{5}$.

   (b) **[10 points]** Find the conditional expectation $\mathsf{E}[\mathbb{Z} \mid \mathbb{N} = 3]$.

   Solution: Conditioned on $\mathbb{N} = 3$, the first two balls drawn are $\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$ with equal probability $\frac{1}{6}$. Thus, conditioned on $\mathbb{N} = 3$, the sum $\mathbb{Z}$ takes on values 3, 4, 6, and 7 with probability $\frac{1}{6}$ and value 5 with probability $\frac{2}{6}$, and hence $\mathsf{E}[\mathbb{Z} \mid \mathbb{N} = 3] = (3 + 4 + 2 \times 5 + 6 + 7)/6 = 5$.
   Alternatively, let $\mathbb{X}_1$ and $\mathbb{X}_2$ denote the numbers on the first two balls drawn. Then, conditioned on $\mathbb{N} = 3$, $\mathbb{X}_1$ takes on values $\{1, 2, 3, 4\}$ with equal probability $\frac{1}{4}$, and $\mathbb{X}_2$ also takes on values $\{1, 2, 3, 4\}$ with equal probability $\frac{1}{4}$. Hence, $\mathsf{E}[\mathbb{Z} \mid \mathbb{N} = 3] = \mathsf{E}[\mathbb{X}_1 + \mathbb{X}_2 \mid \mathbb{N} = 3] = 2 \times (1 + 2 + 3 + 4)/4 = 5$.

2. **[30 points]** Three coins are in a bag. One has heads on both sides, one has tails on both sides, and the third is a fair coin.

   (a) **[10 points]** First, the instructor takes a coin out of the bag, and flips it twice. What is the probability that the first two coin flips both result in "heads?"
   Solution: Let E_i denote the event that the coin selected by the instructor has exactly i heads on it, for $i = 0, 1, 2$ (so E_2 is the event that the instructor draws the two-headed coin from the bag) and let H_j denote the event that heads shows on the j^{th} toss of the instructor. We want to determine $P(H_1H_2)$. Using the law of total probability, we find

$$\begin{aligned} P(H_1H_2) &= P(H_1H_2|E_0)P(E_0) + P(H_1H_2|E_1)P(E_1) + P(H_1H_2|E_2)P(E_2) \\ &= 0 \cdot \frac{1}{3} + (\frac{1}{2})^2 \cdot \frac{1}{3} + 1^2 \cdot \frac{1}{3} = \frac{5}{12}. \end{aligned}$$

   (b) **[10 points]** Given that the first coin flip results in heads, what is the conditional probability that the second flip of the same coin is also heads?
   Solution: $P(H_2|H_1) = \frac{P(H_1H_2)}{P(H_1)} = \frac{5/12}{1/2} = \frac{5}{6}$. (Here we used the fact $P(H_1) = 0.5$. This can be seen by symmetry, or by the method used in part (a).)

   (c) **[10 points]** Suppose "heads" shows on each of the first two coin flips, and then a student selects one of the other two coins in the bag and flips it. What is the conditional probability that the coin flip of the student also shows heads?
   Solution: Let F be the event that the coin flipped by the student shows heads. We will need to compute $P(H_1H_2F)$. There are six possibilities for which coins are selected by instructor and student, each having probability $\frac{1}{6}$. Given the instructor selects the two headed coin and the student selects the fair coin, the conditional probability of H_1H_2F is 0.5. Given the instructor selects the fair coin and the student selects the two headed coin, the conditional probability of H_1H_2F is $(0.5)^2 = 0.25$. Given any of the other four possibilities for which coins were selected by instructor and student, the conditional probability of H_1H_2F is zero. So, $P(H_1H_2F) = (0.5 + 0.25)/6 = \frac{1}{8}$. Therefore, $P(F|H_1H_2) = \frac{P(H_1H_2F)}{P(H_1H_2)} = \frac{1/8}{5/12} = 0.3$.

3. **[16 points]** Suppose $\mathbb{Y} = \mathbb{X}^2$ where $\mathbb{X}$ has pdf $f_{\mathbb{X}}(u) = \begin{cases} 2u, & \text{if } 0 \leq u \leq 1, \\ 0, & \text{otherwise.} \end{cases}$

   Find the CDF of $\mathbb{Y}$.

   Solution: The range of $\mathbb{Y}$ is the interval $[0, 1]$. For c in that interval,
   $F_{\mathbb{Y}}(c) = P\{\mathbb{X}^2 \leq c\} = P\{\mathbb{X} \leq \sqrt{c}\} = \int_0^{\sqrt{c}} 2u du = c$. Of course, $F_{\mathbb{Y}}(c) = 0$ for $c \leq 0$ and $F_{\mathbb{Y}}(c) = 1$ for $c \geq 1$. So $\mathbb{Y}$ is uniformly distributed on the interval $[0, 1]$.

4. **[20 points]** Let $(\mathbb{N}(t) : t > 0)$ denote a Poisson process with arrival rate $\lambda > 0$. Remember that this notation means that for each $t > 0$, the number of arrivals in the interval $(0, t]$ is denoted by $\mathbb{N}(t)$.

   (a) **[10 points]** Find $P\{\mathbb{N}(1) = 2, \mathbb{N}(3) = 5\}$. Note that $\mathbb{N}(1)$ and $\mathbb{N}(3)$ are *not* independent. Your answer should depend on λ.
   Solution:

$$
\begin{aligned}
P\{\mathbb{N}(1) = 2, \mathbb{N}(3) = 5\}, &= P\{\mathbb{N}(1) = 2, \mathbb{N}(3) - \mathbb{N}(1) = 3\}, \\
&= P\{\mathbb{N}(1) = 2\}P\{\mathbb{N}(3) - \mathbb{N}(1) = 3\}, && \text{$\mathbb{N}(1)$ and $\mathbb{N}(3)-\mathbb{N}(1)$ are independent} \\
&= \left(\frac{\lambda^2 e^{-\lambda}}{2!}\right)\left(\frac{(2\lambda)^3 e^{-2\lambda}}{3!}\right) && \text{Poisson RVs with parameters λ,2λ resp.}
\end{aligned}
$$

   (b) **[10 points]** Find $P(\mathbb{N}(1) = 2 \mid \mathbb{N}(3) = 5)$.
   Solution:

$$
\begin{aligned}
P\{\mathbb{N}(1) = 2 \mid \mathbb{N}(3) = 5\} &= \frac{\text{(Answer to part (a))}}{P\{\mathbb{N}(3) = 5\}} \\
&= \frac{\left(\frac{2\lambda^5 e^{-3\lambda}}{3}\right)}{\left(\frac{(3\lambda)^5 e^{-3\lambda}}{5!}\right)} = \frac{80}{243}
\end{aligned}
$$

5. **[20 points]** Suppose $\mathbb{X}$ is a continuous random variable with mean 20 and variance 3. Find the numerical value of $P\{\mathbb{X} < 20 - 1.732\sqrt{3}\}$ (or, nearly equivalently, $P\{\mathbb{X} < 17\}$) in the following two cases:

   (a) **[10 points]** $\mathbb{X}$ is a Gaussian random variable.

   Solution: $P\{\mathbb{X} < 20 - 1.732\sqrt{3}\} = \Phi\left(\frac{20 - 1.732\sqrt{3} - 20}{\sqrt{3}}\right) = \Phi(-1.732) = 1 - \Phi(1.732)$
   $\approx 1 - \Phi(1.73) = 1 - 0.9582 = 0.0418$ from the table supplied.

   (b) **[10 points]** $\mathbb{X}$ is a uniformly distributed random variable.

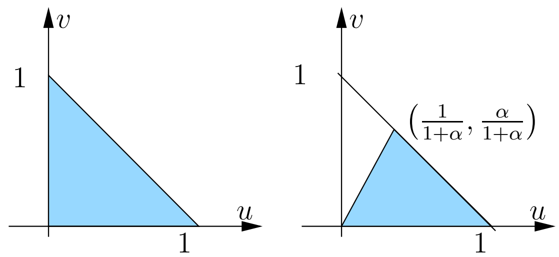
   Solution: If $\mathbb{X}$ is uniformly distributed on $[\alpha, \beta]$, then $\mathsf{E}[\mathbb{X}] = \frac{\beta+\alpha}{2}$ and $\mathsf{var}(\mathbb{X}) = \frac{(\beta-\alpha)^2}{12}$. We readily get $(\beta - \alpha)^2 = 36 \Rightarrow \beta - \alpha = 6$ and $\beta + \alpha = 40$ so that $\alpha = 17$ and $\beta = 23$. Hence, $P\{\mathbb{X} < 20 - 1.732\sqrt{3}\} \approx P\{\mathbb{X} < 17\} = 0$. Since $1.732\sqrt{3}$ is *slightly* smaller than 3, the exact value of $P\{\mathbb{X} < 20 - 1.732\sqrt{3}\}$ is *slightly* larger than 0.

6. **[25 points]** $\mathbb{X}$ and $\mathbb{Y}$ are jointly continuous random variables with joint pdf given by

$$
f_{\mathbb{X},\mathbb{Y}}(u, v) = \begin{cases} 2, & \text{if } u \geq 0,\ v \geq 0,\ u + v \leq 1, \\ 0, & \text{elsewhere.} \end{cases}
$$

   Find the pdf $f_{\mathbb{Z}}(\alpha)$ of the random variable $\mathbb{Z} = \mathbb{Y}/\mathbb{X}$. To obtain full credit, you must specify the value of $f_{\mathbb{Z}}(\alpha)$ for all α, $-\infty < \alpha < \infty$.

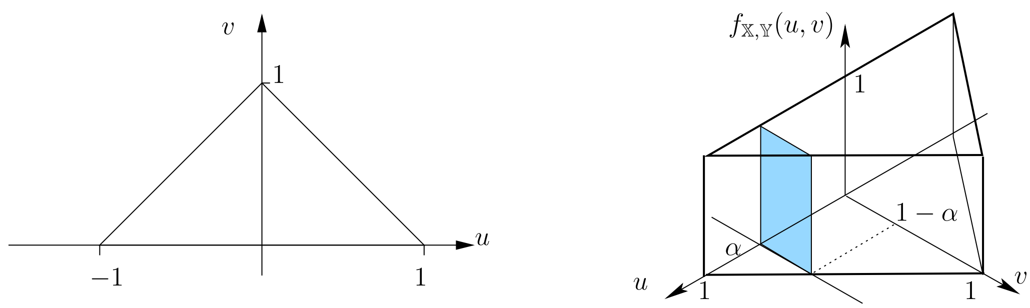
   Solution: The joint pdf has value 2 on the triangular region with vertices at $(0, 0)$, $(0, 1)$, and $(1, 0)$ as shown in the left-hand figure below.



Clearly, $\mathbb{Z} = \mathbb{Y}/\mathbb{X}$ takes on values in $[0, \infty)$ and so $f_{\mathbb{Z}}(\alpha) = 0$ for $\alpha < 0$. For any real number α such that $0 \leq \alpha < \infty$, we have $F_{\mathbb{Z}}(\alpha) = P\{\mathbb{Z} \leq \alpha\} = P\{\mathbb{Y}/\mathbb{X} \leq \alpha\}$
$= P\{$random point $(\mathbb{X}, \mathbb{Y})$ lies in shaded region shown in right-hand figure above$\}$
$= \int\int f_{\mathbb{X},\mathbb{Y}}(u, v)\, du\, dv$ over the shaded region in right-hand figure above $= 2 \times \{$area of region$\}$
$= 2\times\left(\frac{1}{2}\times 1\times \frac{\alpha}{\alpha+1}\right) = \frac{\alpha}{\alpha+1}$ since the shaded region is a triangle of base 1 and apex at $(1/(\alpha+1), \alpha/(\alpha+1))$.
It follows that $f_{\mathbb{Z}}(\alpha) = \frac{d}{d\alpha} F_{\mathbb{Z}}(\alpha) = \frac{1}{(1+\alpha)^2}$ for $\alpha > 0$.

7. **[36 points]** The joint pdf of the random variables $\mathbb{X}$ and $\mathbb{Y}$ has constant value 1 on the triangular region with vertices at $(-1, 0)$, $(0, 1)$, and $(1, 0)$.

   The pdf is nonzero on the triangular region shown in the left-hand figure below and defines a triangular prism as shown in the right-hand figure below.



(a) **[8 points]** Find the value of $\mathsf{E}[\mathbb{Y}]$.

Solution: $\mathsf{E}[\mathbb{Y}] = \displaystyle\int_{v=0}^{1}\int_{u=-(1-v)}^{1-v} v\cdot 1\, du\, dv = \int_{v=0}^{1} 2v(1-v)\, dv = v^2 - \frac{2v^3}{3}\Big|_0^1 = \frac{1}{3}.$

(b) **[6 points]** Find $f_{\mathbb{Y}|\mathbb{X}}(v \mid \frac{\pi}{4})$, the conditional pdf of $\mathbb{Y}$ given that $\mathbb{X} = \pi/4$.

Solution: The conditional pdf of $\mathbb{Y}$ given $\mathbb{X} = \alpha$ is the cross-section of the joint pdf solid at α normalized to have unit area. The cross-section is a rectangle with base $1 - \alpha$ as indicated in the right-hand figure above. (More generally, the rectangle base is $1 - |\alpha|$ since α might have negative value). We see that given $\mathbb{X} = \pi/4$, the conditional pdf of $\mathbb{Y}$ is a uniform density on $[0, 1 - \pi/4]$.

More formally, $f_{\mathbb{Y}|\mathbb{X}}(v \mid \frac{\pi}{4}) = \begin{cases} \frac{4}{4-\pi}, & \text{if } 0 \leq v \leq 1 - \frac{\pi}{4}, \\ 0, & \text{otherwise.} \end{cases}$

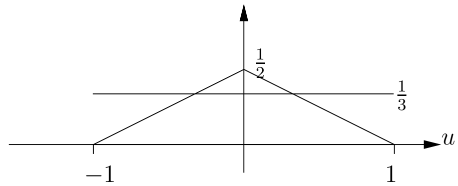
(c) **[6 points]** Find $\mathsf{E}[\mathbb{Y} \mid \mathbb{X} = \frac{\pi}{4}]$, the conditional mean of $\mathbb{Y}$ given that $\mathbb{X} = \pi/4$.

Solution: The mean of a uniform random variable is the midpoint of its range. Hence we get that $\mathsf{E}[\mathbb{Y} \mid \mathbb{X} = \frac{\pi}{4}] = \frac{1}{2} - \frac{\pi}{8}$.

(d) **[8 points]** Sketch, as a function of u, a graph of the *minimum mean-square error estimator* of $\mathbb{Y}$ given that value of $\mathbb{X}$ is u, for u in the range $u \in (-1, 1)$.

Solution: The MMSE estimator of $\mathbb{Y}$ given $\mathbb{X} = u$ is the condiitonal mean of $\mathbb{Y}$ for the given value of $\mathbb{X}$. Since the conditional pdf is uniform on $1 - |u|$, the conditional mean is $(1 - |u|)/2$ and is sketched in the figure below.

MMSE estimators for $\mathbb{Y}$



(e) **[8 points]** Sketch a graph of the **linear** *minimum mean-square error estimator* of $\mathbb{Y}$ given that value of $\mathbb{X}$ is u, where $u \in (-1, 1)$.
Hint: Without doing any actual integrations, first deduce that $\mathsf{E}[\mathbb{X}\mathbb{Y}] = \mathsf{E}[\mathbb{X}] = 0$.
Solution: The *linear* MMSE estimator for $\mathbb{Y}$ given that $\mathbb{X} = u$ is $\mu_{\mathbb{Y}} + \rho\frac{\sigma_{\mathbb{Y}}}{\sigma_{\mathbb{X}}}(u - \mu_{\mathbb{X}})$. In part (a), we found that $\mu_{\mathbb{Y}} = \frac{1}{3}$. It is obvious from the symmetry of the pdf that $\mathsf{E}[\mathbb{X}] = 0$ but anti-segregationists can use $\mathsf{E}[\mathbb{X}] = \int_{v=0}^{1}\int_{u=-(1-v)}^{1-v} u \cdot 1\, du\, dv = \int_{v=0}^{1} \frac{u^2}{2}\Big|_{-(1-v)}^{1-v} dv = \int_0^1 0\, dv = 0$.
Similarly, $\mathsf{E}[\mathbb{X}\mathbb{Y}] = \int_{v=0}^{1}\int_{u=-(1-v)}^{1-v} uv \cdot 1\, du\, dv = \int_{v=0}^{1} v\frac{u^2}{2}\Big|_{-(1-v)}^{1-v} dv = \int_0^1 0\, dv = 0$ giving us that $\mathsf{cov}(\mathbb{X}, \mathbb{Y}) = \mathsf{E}[\mathbb{X}\mathbb{Y}] - \mathsf{E}[\mathbb{X}]\mathsf{E}[\mathbb{Y}] = 0 \Rightarrow rho = 0$. Hence, the linear MMSE estimator for $\mathbb{Y}$ is $mu_{\mathbb{Y}} = \frac{1}{3}$. This is illustrated in the figure above.

8. **[28 points]** Suppose $\mathbb{X}$ and $\mathbb{Y}$ are *zero-mean unit-variance* jointly Gaussian random variables with correlation coefficient $\rho = 0.5$.

(a) **[8 points]** Find $\mathsf{var}(3\mathbb{X} - 2\mathbb{Y})$.
Solution: $\mathsf{var}(3\mathbb{X} - 2\mathbb{Y}) = 3^2 \cdot \mathsf{var}(\mathbb{X}) + 2^2 \cdot \mathsf{var}(\mathbb{Y}) - 2 \cdot 3 \cdot 2 \cdot \mathsf{cov}(\mathbb{X}, \mathbb{Y}) = 9 + 4 - 12 \times \frac{1}{2} = 7$.

(b) **[10 points]** Find the numerical value of $P\{(3\mathbb{X} - 2\mathbb{Y})^2 \leq 28\}$.
Solution: $\mathsf{E}[3\mathbb{X} - 2\mathbb{Y}] = 3 \cdot \mathsf{E}[\mathbb{X}] - 2 \cdot \mathsf{E}[\mathbb{Y}] = 0$. Furthermore, since $\mathbb{X}$ and $\mathbb{Y}$ are *jointly Gaussian* random variables, $3\mathbb{X} - 2\mathbb{Y}$ is also a Gaussian random variable, and we have that
$P\{(3\mathbb{X} - 2\mathbb{Y})^2 \leq 28\} = P\{-\sqrt{28} \leq 3\mathbb{X} - 2\mathbb{Y} \leq \sqrt{28}) = \Phi\left(\frac{\sqrt{28} - 0}{\sqrt{7}}\right) - \Phi\left(-\frac{\sqrt{28} - 0}{\sqrt{7}}\right)$
$= \Phi(2) - \Phi(-2) = \Phi(2) - [1 - \Phi(2)] = 2\Phi(2) - 1 = 2 \times 0.9772 - 1 = 0.9544$.

(c) **[10 points]** Find $\mathsf{E}[\mathbb{Y} \mid \mathbb{X} = 3]$.
Solution: Since $\mathbb{X}$ and $\mathbb{Y}$ are *jointly Gaussian* random variables, the conditional mean of $\mathbb{Y}$ given $\mathbb{X} = \alpha$ is the same as the *linear* MMSE estimator of $\mathbb{Y}$ given $\mathbb{X} = \alpha$, *viz.* $\mu_{\mathbb{Y}} + \rho(\sigma_{\mathbb{Y}}/\sigma_{\mathbb{X}})(\alpha - \mu_{\mathbb{X}})$ $= 0 + 0.5 \times 1 \times (3 - 0) = 3/2$.

9. **[30 points]** (3 points per answer)
In order to discourage guessing, 3 points will be deducted for each incorrect answer (no penalty or gain for blank answers). A net negative score will reduce your total exam score.

(a) A and B are two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$.

TRUE	FALSE	
☐	☐	$P(A \cup B) \geq \max\{P(A), P(B)\}$
☐	☐	$P(A\|B) + P(A\|B^c) = 1$.
☐	☐	$P(A\|B)P(B) + P(A^c\|B)P(B) = P(A)$.
☐	☐	If $P(A) = P(B)$, then $P(A\|B) = P(B\|A)$.

Solution: A and B are two events such that $0 < P(A) < 1$ and $0 < P(B) < 1$.
$P(A \cup B) \geq \max\{P(A), P(B)\}$ is a TRUE statement. $A, B \subset A \cup B \Rightarrow P(A), P(B) \leq P(A \cup B)$.
$P(A|B) + P(A|B^c) = 1$ is a FALSE statement. On the other hand, the very similar-looking statement $P(A|B) + P(A^c|B) = 1$ is TRUE.
$P(A|B)P(B) + P(A^c|B)P(B) = P(A)$ is a FALSE statement. On the other hand, the very similar-looking statement $P(A|B)P(B) + P(A^c|B)P(B) = P(B)$ is TRUE.

If $P(A) = P(B)$, then $P(A|B) = P(B|A)$ is a TRUE statement. On the other hand, the converse statement (found on the Conflict Final Exam) "If $P(A|B) = P(B|A)$, then $P(A) = P(B)$" is FALSE. Can you figure out why?

(b) $\mathbb{X}$ is a continuous random variable whose probability density function (pdf) $f_{\mathbb{X}}(u)$ is an *even function*, that is, $f_{\mathbb{X}}(u) = f_{\mathbb{X}}(-u)$ for *all* real numbers u. Let $F_{\mathbb{X}}(u)$ denote the cumulative probability distribution function (CDF) of $\mathbb{X}$, let α denote a *positive* real number, and assume that that $\mathbb{X}$ has finite variance σ^2.

TRUE	FALSE	
☐	☐	$\mathsf{E}[\mathbb{X}] = 0$.
☐	☐	$F_{\mathbb{X}}(-\alpha) \leq \dfrac{\sigma^2}{2\alpha^2}$.
☐	☐	$P\{\mathbb{X}^2 > \alpha^2\} = 2F_{\mathbb{X}}(-\alpha)$.
☐	☐	The pdf of $\mathbb{Y} = \lvert\mathbb{X}\rvert$ is $f_{\mathbb{Y}}(v) = 2f_{\mathbb{X}}(v) - 1, 0 \leq v < \infty$.

Solution: $f_{\mathbb{X}}(u)$ is an even function and the variance σ^2 is finite.

$\mathsf{E}[\mathbb{X}] = 0$ is a TRUE statement because the pdf is an even function and σ^2 is finite. Note that there are random variables with even pdfs for which the statement $\mathsf{E}[\mathbb{X}] = 0$ is false, for example, the Cauchy random variable.

$F_{\mathbb{X}}(-\alpha) \leq \frac{\sigma^2}{2\alpha^2}$ is a TRUE statement. By the Chebyshev inequality, we have that $P\{|\mathbb{X}| > \alpha\} = F_{\mathbb{X}}(-\alpha) + (1 - F_{\mathbb{X}}(\alpha)) \leq \frac{\sigma^2}{2\alpha^2}$. But because of the pdf is an even function, $F_{\mathbb{X}}(-\alpha) = (1 - F_{\mathbb{X}}(\alpha))$.

$P\{\mathbb{X}^2 > \alpha^2\} = 2F_{\mathbb{X}}(-\alpha)$ is a TRUE statement. Note that $P\{\mathbb{X}^2 > \alpha^2\} = P\{|\mathbb{X}| > \alpha\}$ and look at the previous part of this problem.

"The pdf of $\mathbb{Y} = |\mathbb{X}|$ is $f_{\mathbb{Y}}(v) = 2f_{\mathbb{X}}(v) - 1, 0 \leq v < \infty$" is a FALSE statement. Note that the purported pdf has value close to -1 for large values of v.

(c) $\mathbb{X}$ and $\mathbb{Y}$ are random variables with finite and equal variances, that is, $\mathsf{var}(\mathbb{X}) = \mathsf{var}(\mathbb{Y}) = \sigma^2 < \infty$. Suppose that $\mathsf{var}(2\mathbb{X} + 3\mathbb{Y} + 4) = \mathsf{var}(3\mathbb{X} - 2\mathbb{Y} + 1)$.

TRUE	FALSE	
☐	☐	$\mathbb{X}$ and $\mathbb{Y}$ are *uncorrelated* random variables.
☐	☐	$\mathsf{var}(2\mathbb{X} + 3\mathbb{Y} + 4) = \mathsf{var}(2\mathbb{X} - 3\mathbb{Y} + 1)$.

Solution: $\mathsf{var}(2\mathbb{X} + 3\mathbb{Y} + 4) = 4 \cdot \mathsf{var}(\mathbb{X}) + 9 \cdot \mathsf{var}(\mathbb{Y}) + 2 \cdot 2 \cdot 3 \cdot \mathsf{cov}(\mathbb{X}, \mathbb{Y}) = 13\sigma^2 + 12 \cdot \mathsf{cov}(\mathbb{X}, \mathbb{Y})$.
$\mathsf{var}(3\mathbb{X} - 2\mathbb{Y} + 1) = 9 \cdot \mathsf{var}(\mathbb{X}) + 4 \cdot \mathsf{var}(\mathbb{Y}) - 2 \cdot 3 \cdot 2 \cdot \mathsf{cov}(\mathbb{X}, \mathbb{Y}) = 13\sigma^2 + 12 \cdot \mathsf{cov}(\mathbb{X}, \mathbb{Y})$.
Since these two variances are equal, then it must be that $\mathsf{cov}(\mathbb{X}, \mathbb{Y}) = 0$. Hence, "$\mathbb{X}$ and $\mathbb{Y}$ are *uncorrelated* random variables." is a TRUE statement.

Since additive constants do not affect variance, "$\mathsf{var}(2\mathbb{X} + 3\mathbb{Y} + 4) = \mathsf{var}(2\mathbb{X} - 3\mathbb{Y} + 1)$" is a TRUE statement.