University of Illinois Spring 2009

ECE 313: Hour Exam I

Monday March 2, 2009
7:00 p.m. — 8:00 p.m.
100 Noyes Laboratory

1. (a) The hat contains balls $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{2,3\}$, $\{2,4\}$, $\{3,4\}$ with equal probability $\frac{1}{6}$. Hence, $P(3 \text{ ball is in the hat}) = \frac{1}{2}$.

   (b) From the answer to part (a), we can read off the values of the pmf: $p_{\mathbb{X}}(3) = \frac{1}{6}$, $p_{\mathbb{X}}(4) = \frac{1}{6}$, $p_{\mathbb{X}}(5) = \frac{2}{6} = \frac{1}{3}$, $p_{\mathbb{X}}(6) = \frac{1}{6}$, $p_{\mathbb{X}}(3) = \frac{1}{6}$, and $p_{\mathbb{X}}(u) = 0$ for all other values of u. Sanity check: $p_{\mathbb{X}}(3) + p_{\mathbb{X}}(4) + p_{\mathbb{X}}(5) + p_{\mathbb{X}}(6) + p_{\mathbb{X}}(7) = 1$ as it should be.

   (c) Following the theft of the 3 ball, the hat contains $\{1,2\}$, $\{1\}$, $\{1,4\}$, $\{2\}$, $\{2,4\}$, $\{4\}$ with equal probability $\frac{1}{6}$. Hence, by the law of total probability,

$$P(\text{ball drawn has an odd number}) = \left[\frac{1}{2} + 1 + \frac{1}{2} + 0 + 0 + 0\right] \times \frac{1}{6} = \frac{1}{3}.$$

2. (a) $P(\text{Fred wins game}) = P\{H, TTH, TTTTH, \ldots\} = P\{H\} + P\{TTH\} + P\{TTTTH\} + \cdots = \frac{1}{2} + \left(\frac{1}{2}\right)^2 \frac{1}{2} + \left(\frac{1}{2}\right)^4 \frac{1}{2} + \cdots = \frac{1}{2}\left[1 + \frac{1}{4} + \left(\frac{1}{4}\right)^2 + \cdots\right] = \frac{1}{2} \times \frac{1}{1 - 1/4} = \frac{4}{2 \times 3} = \frac{2}{3}$.
   Similarly, $P(\text{Wilma wins first game}) = \dfrac{1}{3}$.
   Alternatively, with F and W denoting the events that Fred and Wilma respectively win the first game, and H the event that the first toss (by Fred) results in a Head, we have $P(F \mid H) = 1$ and $P(W \mid H) = 0$). Also, if $T = H^c$ occurs on the first toss, then Wilma gets to toss in a brand new game and thus $P(F \mid T) = P(W)$, $P(W \mid T) = P(F)$. By the law of total probability, we get

$$P(F) = P(F \mid H)P(H) + P(F \mid T)P(T) = 1 \times \frac{1}{2} + P(W) \times \frac{1}{2} = \frac{1}{2} + \frac{P(W)}{2},$$
$$P(W) = P(W \mid H)P(H) + P(W \mid T)P(T) = 0 \times \frac{1}{2} + P(F) \times \frac{1}{2} = \frac{P(F)}{2}.$$

   It follows that $P(F) = \dfrac{1}{2} + \dfrac{1}{4} \times P(F)$ and hence $P(F) = \dfrac{2}{3}$ and $P(W) = \dfrac{1}{3}$.

   (b) Let F and W denote the events that Fred and Wilma respectively win a game, and let FFW denote the event that Fred wins the first two games and Wilma the third, etc. Remembering that the winner of a game has only a $\frac{1}{3}$ chance of winning the next game, we proceed to calculate systematically as follows:

Result	Value of $\mathbb{X}$	Probability	Result	Value of $\mathbb{X}$	Probability
FFF	-3	$\frac{2}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{2}{27}$	WWF	$+1$	$\frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{2}{27}$
FFW	-1	$\frac{2}{3} \times \frac{1}{3} \times \frac{2}{3} = \frac{4}{27}$	WFW	$+1$	$\frac{1}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{4}{27}$
FWF	-1	$\frac{2}{3} \times \frac{2}{3} \times \frac{2}{3} = \frac{8}{27}$	FWW	$+1$	$\frac{2}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{4}{27}$
WFF	-1	$\frac{1}{3} \times \frac{2}{3} \times \frac{1}{3} = \frac{2}{27}$	WWW	$+3$	$\frac{1}{3} \times \frac{1}{3} \times \frac{1}{3} = \frac{1}{27}$

Hence, $p_{\mathbb{X}}(-3) = \dfrac{2}{27}$, $p_{\mathbb{X}}(-1) = \dfrac{14}{27}$, $p_{\mathbb{X}}(1) = \dfrac{10}{27}$, $p_{\mathbb{X}}(3) = \dfrac{1}{27}$, and $p_{\mathbb{X}}(u) = 0$ for all other values of u. Sanity check: $p_{\mathbb{X}}(-3) + p_{\mathbb{X}}(-1) + p_{\mathbb{X}}(1) + p_{\mathbb{X}}(3) = 1$ as it should be.

3. (a) Let $C = A \cup B$ denote the event that at least one of the events A and B occurs. Let $D = A^c \cup B^c$ denote the event that at least one of A and B does not occur. We are given that $P(C) = P(D) = 0.8$. Note that $C \cup D = \Omega$ has probability 1 while $CD = (A \cup B)(a^c \cup B^c) = AB^c \cup A^cB$ is precisely the event whose probability we want. Thus,
$1 = P(C \cup D) = P(C) + P(D) - P(CD) \Rightarrow P(CD) = P(AB^c \cup A^cB) = 0.8 + 0.8 - 1 = 0.6$

   Alternatively, by DeMorgan's theorem,
   $P(\text{neither } A \text{ nor } B \text{ occurs}) = P(A^cB^c) = P((A \cup B)^c) = 1 - P(A \cup B) = 0.2$.
   Similarly, $P(\text{both } A \text{ and } B \text{ occur}) = P(AB) = P((A^c \cup B^c)^c) = 1 - P(A^c \cup B^c) = 0.2$.
   Hence, $P(\text{exactly one of } A \text{ and } B \text{ occurs}) = 1 - P(\text{both occur}) - P(\text{neither occurs}) = 0.6$ as before.

   (b) $(n, p) = (100, 0.2)$. $\mathsf{E}[\mathbb{X}] = np = 100 \times 0.2 = 20$. $\mathsf{var}(\mathbb{X}) = np(1-p) = 100 \times 0.2 \times 0.8 = 16$. Hence, $\mathsf{E}[\mathbb{X}^2] = \mathsf{var}(\mathbb{X}) + (\mathsf{E}[\mathbb{X}])^2 = 16 + 20^2 = 416$.

4. (a) The probability that a particular square is selected is just the probability of picking that row ($\frac{1}{3}$) times the conditional probability of picking that square among the 2, 3, or 4 squares on the chosen row. Hence, we get

a	$\frac{1}{6}$	$\frac{1}{6}$		
b	$\frac{1}{9}$	$\frac{1}{9}$	$\frac{1}{9}$	
c	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{12}$
	1	2	3	4

For $i \in \{1, 2, 3\}$, let R_i be the event that the square selected is in the i-th *row*.
For $j \in \{1, 2, 3, 4\}$, let C_j be the event that the square selected is in the j-th *column*..

   (b) $P(C_2) = \frac{1}{6} + \frac{1}{9} + \frac{1}{12} = \frac{13}{36}$.

   (c) $P(R_1 \mid C_2) = \frac{1/6}{13/36} = \frac{6}{13}$.

   (d) $P(C_2 \mid R_2) = \frac{1/9}{1/3} = \frac{1}{3}$. We also know the answer is $\frac{1}{3}$ directly from the problem statement: given row b is selected, each square in the row has equal probability.