University of Illinois Spring 2009

ECE 313: Hour Exam II

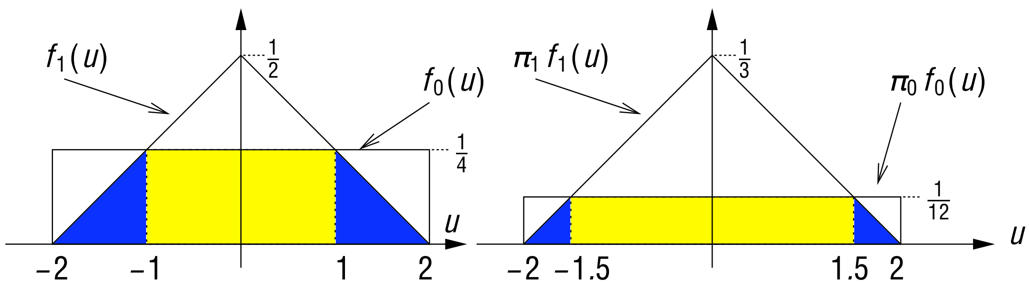
Monday April 13, 2009
7:00 p.m. — 8:00 p.m.
100 Noyes Laboratory

1. (a) By DeMorgan's law, $A^c \cup B^c \cup C^c = (ABC)^c$. Hence, $P(A^c \cup B^c \cup C^c) = 1 - P(ABC)$, and since A, B, C are mutually independent, we have $P(ABC) = P(A)P(B)P(C) = 0.1 \times 0.2 \times 0.2 = 0.004$ and $P(A^c \cup B^c \cup C^c) = 1 - P(ABC) = 0.996$.

   (b) As in part (a), $P(A^c \cup B^c \cup C^c) = 1 - P(ABC)$. But now $P(ABC) = 0$ since A, B, C are mutually exclusive events. Hence, $P(A^c \cup B^c \cup C^c) = 1 - P(ABC) = 1$.

   (c) Since $AB = \emptyset$ and BC is a subset of B, we get that $A(BC) = (AB)C = \emptyset$ and so $P(A \cup BC) = P(A) + P(BC)$. But $P(BC) = P(B)P(C)$ since B and C are independent events, and so $P(A \cup BC) = P(A) + P(BC) = 0.1 + 0.2 \times 0.2 = 0.14$.

2. (a) The easiest way to solve this problem is to sketch the two pdfs as shown in the left-hand figure below.



   It is obvious that the maximum-likelihood decision is in favor of H_1 if $|\mathbb{X}| < 1$, and hence $x = 0,\ \eta = 1$. By inspection, we get that $P_{\text{FA}} = 2 \times \frac{1}{4} = \frac{1}{2}$ while $P_{\text{MD}} = 2 \times \left(\frac{1}{2} \times 1 \times \frac{1}{4}\right) = \frac{1}{4}$.

   The graphically-challenged can proceed as follows.
   For $-2 < u < 2$, the likelihood ratio is $\Lambda(u) = \dfrac{f_1(u)}{f_0(u)} = \dfrac{0.25(2 - |u|)}{0.25} = 2 - |u|$.
   When $\mathbb{X} = u$ is the observation, the *maximum-likelihood* decision rule decides in favor of H_1 if $\Lambda(u) > 1$. Hence $\Gamma_1 = \{u \ :\ |u| < 1\}$ and $\Gamma_0 = \{u \ :\ 1 < |u| < 2\}$, that is, the ML decision rule is that if $|\mathbb{X}| > 1$, the decision is that H_0 is the true hypothesis. Thus, we have $x = 0$, and $\eta = 1$.
   $P_{\text{FA}} = \displaystyle\int_{\Gamma_1} f_0(u)\,du = \int_{-1}^{1} \frac{1}{4} du = \frac{1}{2}.$
   $P_{\text{MD}} = \displaystyle\int_{\Gamma_0} f_1(u)\,du = 2\int_{1}^{2} \frac{1}{4}(2-u)\,du = \frac{1}{2}\left(2u - \frac{u^2}{2}\right)\Bigg|_1^2 = \frac{1}{4}.$

   (b) The probability of error of the ML decision rule is

$$P(E) = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{4} = \frac{1}{3}.$$

   (c) Sketching $\pi_0 f_0(u)$ and $\pi_1 f_1(u)$ as in the right-hand figure above, we easily see that the MAP decision is in favor of H_1 if $|\mathbb{X}| < 1.5$, and hence $x = 0,\ \xi = 1.5$. By inspection, we get that $\pi_0 P_{\text{FA}} = 3 \times \frac{1}{12} = \frac{1}{4} = \frac{1}{3} \times \frac{3}{4}$ while $\pi_1 P_{\text{MD}} = 2 \times \left(\frac{1}{2} \times \frac{1}{2} \times \frac{1}{12}\right) = \frac{1}{24} = \frac{2}{3} \times \frac{1}{16}$, that is, $P_{\text{FA}} = \frac{3}{4}$, and $P_{\text{MD}} = \frac{1}{16}$. $P(E) = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = \frac{1}{4} + \frac{1}{24} = \frac{7}{24} < \frac{1}{3}$, where $\frac{1}{3}$ is the error probability of the ML rule (with the same *a priori* probabilities) that we found in part (c).

   Without using any graphical aids, we have that when $\mathbb{X} = u$ is the observation, the MAP decision rule decides in favor of H_1 if $\Lambda(u) = 2 - |u| > \pi_0/\pi_1 = 1/2$. Hence, $\Gamma_1 = \left\{u \ :\ |u| < \frac{3}{2}\right\}$ and

$\Gamma_0 = \left\{u : \frac{3}{2} < |u| < 2\right\}$ for the MAP decision rule. Once again, $x = 0$ while $\xi = \frac{3}{2}$. We get
$P_{\text{FA}} = \int_{\Gamma_1} f_0(u)\,du = \int_{-\frac{3}{2}}^{\frac{3}{2}} \frac{1}{4} du = \frac{3}{4}.$
$P_{\text{MD}} = \int_{\Gamma_0} f_1(u)\,du = 2\int_{\frac{3}{2}}^{2} \frac{1}{4}(2-u)\,du = \frac{1}{2}\left(2u - \frac{u^2}{2}\right)\Big|_{\frac{3}{2}}^{2} = \frac{1}{16}.$
Hence, $P(E) = \pi_0 \cdot P_{\text{FA}} + \pi_1 \cdot P_{\text{MD}} = \frac{1}{3} \times \frac{3}{4} + \frac{2}{3} \times \frac{1}{16} = \frac{1}{4} + \frac{1}{24} = \frac{7}{24} < \frac{1}{3}.$

3. (a) The pdf f_X is K times the $N(\mu = 2, \sigma^2 = 4)$ pdf truncated to the interval $[0,4]$. Let Z be a $N(2,4)$ random variable. Then $P\{-1 \le X \le 1\} = P\{0 \le X \le 1\}$
$= P\{0 \le Z \le 1\}K = P\{-1 \le \frac{Z-2}{2} \le \frac{1-2}{2}\}K = (\Phi(-0.5) - \Phi(-1))K$
$= (\Phi(1) - \Phi(0.5))K = (0.8413 - 0.6915)K = (0.1498)K.$

   (b) With Z as in the solution to part (a) we have

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} f_X(x)dx = P\{0 \le Z \le 4\}K \\ &= P\left\{\frac{0-2}{2} \le \frac{Z-2}{2} \le \frac{4-2}{2}\right\}K = \Phi(1) - \Phi(-1) = 2(\Phi(1) - 0.5)K \\ &= 2(0.8413 - 0.5)K = (0.6826)K,\end{aligned}$$

   so we have $K = 1/0.6826 \approx 1.46$.

   (c) Since the pdf $f_X(u)$ is symmetric about the point $u = 2$, and has bounded support (so the mean exists), it follows that $E[X] = 2$.

4. (a) Since X ranges over the interval $[0,1]$, Y ranges over $[1,32]$. For $1 \le c \le 32$, $P\{Y \le c\} = P\{(1+X)^5 \le c\} = P\{1 + X \le c^{\frac{1}{5}}\} = P\{X \le c^{\frac{1}{5}} - 1\} = c^{\frac{1}{5}} - 1$. So,

$$F_Y(c) = \begin{cases} 0 & c \le 1 \\ c^{\frac{1}{5}} - 1 & 1 \le c \le 32 \\ 1 & c \ge 32. \end{cases}$$

   (b) Since f_Y is the derivative of the CDF found in part (a),

$$f_Y(c) = \begin{cases} \frac{1}{5}c^{-\frac{4}{5}} & 1 \le c \le 32 \\ 0 & \text{else} \end{cases}$$

   (c) By LOTUS, $E[Y] = E[(1+X)^5] = \int_0^1 (1+u)^5 \cdot 1\,du = \frac{1}{6}(1+u)^6\Big|_0^1 = \frac{2^6 - 1}{6}$
$= \frac{21}{2}$. Alternatively, $E[Y] = \int_1^{32} c \cdot \frac{1}{5}c^{-\frac{4}{5}}\,dc = \int_1^{32} \frac{1}{5}c^{\frac{1}{5}}\,dc = \frac{1}{6}c^{\frac{6}{5}}\Big|_0^{32} = \frac{(2^5)^{\frac{6}{5}} - 1}{6} = \frac{21}{2}.$