1. (a) By DeMorgan’s law, \(A^c \cup B^c \cup C^c = (ABC)^c\). Hence, \(P(A^c \cup B^c \cup C^c) = 1 - P(ABC)\), and since \(A, B, C\) are mutually independent, we have \(P(ABC) = P(A)P(B)P(C) = 0.1 \times 0.2 \times 0.2 = 0.004\) and \(P(A^c \cup B^c \cup C^c) = 1 - P(ABC) = 0.996\).

(b) As in part (a), \(P(A^c \cup B^c \cup C^c) = 1 - P(ABC)\). But now \(P(ABC) = 0\) since \(A, B, C\) are mutually exclusive events. Hence, \(P(A^c \cup B^c \cup C^c) = 1 - P(ABC) = 1\).

(c) Since \(AB = \emptyset\) and \(BC\) is a subset of \(B\), we get that \(A(BC) = (AB)C = \emptyset\) and so \(P(A \cup BC) = P(A) + P(BC)\). But \(P(BC) = P(B)P(C)\) since \(B\) and \(C\) are independent events, and so \(P(A \cup BC) = P(A) + P(BC) = 0.1 + 0.2 \times 0.2 = 0.14\).

2. (a) The easiest way to solve this problem is to sketch the two pdfs as shown in the left-hand figure below.

It is obvious that the maximum-likelihood decision is in favor of \(H_1\) if \(|X| < 1\), and hence \(x = 0\), \(\eta = 1\). By inspection, we get that \(P_{\text{FA}} = 2 \times \frac{1}{2} = \frac{1}{2}\) while \(P_{\text{MD}} = 2 \times \left(\frac{1}{2} \times 1 \times \frac{1}{4}\right) = \frac{1}{4}\).

The graphically-challenged can proceed as follows.

For \(-2 < u < 2\), the likelihood ratio is \(\Lambda(u) = \frac{f_1(u)}{f_0(u)} = \frac{0.25(2 - |u|)}{0.25} = 2 - |u|\).

When \(X = u\) is the observation, the maximum-likelihood decision rule decides in favor of \(H_1\) if \(\Lambda(u) > 1\). Hence \(\Gamma_1 = \{u : |u| < 1\}\) and \(\Gamma_0 = \{u : 1 < |u| < 2\}\), that is, the ML decision rule is that if \(|X| > 1\), the decision is that \(H_0\) is the true hypothesis. Thus, we have \(x = 0\), and \(\eta = 1\).

\[
P_{\text{FA}} = \int_{\Gamma_1} f_0(u) \, du = \int_{-1}^{1} \frac{1}{4} \, du = \frac{1}{2}
\]

\[
P_{\text{MD}} = \int_{\Gamma_0} f_1(u) \, du = 2 \int_{1}^{2} \frac{1}{4} (2 - u) \, du = \frac{1}{2} \left(2u - \frac{u^2}{2}\right)\bigg|_{1}^{2} = \frac{1}{4}.
\]

(b) The probability of error of the ML decision rule is

\[
P(E) = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = \frac{1}{3} \times \frac{1}{2} + \frac{2}{3} \times \frac{1}{4} = \frac{1}{3}.
\]

(c) Sketching \(\pi_0 f_0(u)\) and \(\pi_1 f_1(u)\) as in the right-hand figure above, we easily see that the MAP decision is in favor of \(H_1\) if \(|X| < 1.5\), and hence \(x = 0\), \(\xi = 1.5\). By inspection, we get that \(\pi_0 P_{\text{FA}} = 3 \times \frac{1}{12} = \frac{1}{4} = \frac{1}{3} \times \frac{1}{4}\) while \(\pi_1 P_{\text{MD}} = 2 \times \left(\frac{1}{2} \times \frac{3}{4} \times \frac{1}{10}\right) = \frac{1}{24} = \frac{2}{3} \times \frac{1}{12}\), that is, \(P_{\text{FA}} = \frac{3}{4}\), and \(P_{\text{MD}} = \frac{1}{12}\). \(P(E) = \pi_0 P_{\text{FA}} + \pi_1 P_{\text{MD}} = \frac{1}{3} + \frac{1}{24} = \frac{7}{24} < \frac{1}{3}\), where \(\frac{1}{3}\) is the error probability of the ML rule (with the same a priori probabilities) that we found in part (c).

Without using any graphical aids, we have that when \(X = u\) is the observation, the MAP decision rule decides in favor of \(H_1\) if \(\Lambda(u) = 2 - |u| > \pi_0/\pi_1 = 1/2\). Hence, \(\Gamma_1 = \{u : |u| < \frac{3}{2}\}\) and
3. (a) The pdf \( f_X \) is \( K \) times the \( N(\mu = 2, \sigma^2 = 4) \) pdf truncated to the interval \([0, 4]\). Let \( Z \) be a \( N(2, 4) \) random variable. Then

\[ P\{-1 \leq X \leq 1\} = P\{0 \leq X \leq 1\} = P\{0 \leq Z \leq 1\} = P\{-1 \leq \frac{Z-2}{2} \leq \frac{1-2}{2}\} = (\Phi(-0.5) - \Phi(-1))K = (\Phi(1) - \Phi(0.5))K = (0.8413 - 0.6915)K = (0.1498)K. \]

(b) With \( Z \) as in the solution to part (a) we have

\[ \begin{align*} 1 &= \int_{-\infty}^{\infty} f_X(x)dx = P\{0 \leq Z \leq 4\}K \\
&= P\left\{ \frac{0-2}{2} \leq \frac{Z-2}{2} \leq \frac{4-2}{2} \right\} = \Phi(1) - \Phi(-1) = 2(\Phi(1) - 0.5)K \\
&= 2(0.8413 - 0.5)K = (0.6826)K, \end{align*} \]

so we have \( K = 1/0.6826 \approx 1.46. \)

(c) Since the pdf \( f_X(u) \) is symmetric about the point \( u = 2 \), and has bounded support (so the mean exists), it follows that \( E[X] = 2. \)

4. (a) Since \( X \) ranges over the interval \([0, 1]\), \( Y \) ranges over \([1,32]\). For \( 1 \leq c \leq 32 \), \( P\{Y \leq c\} = P\{(1 + X)^5 \leq c\} = P\{1 + X \leq c^{1/5}\} = P\{X \leq c^{5/5} - 1\} = c^{5/5} - 1. \) So,

\[ F_Y(c) = \begin{cases} 0 & c \leq 1 \\ c^{1/5} - 1 & 1 \leq c \leq 32 \\ 1 & c \geq 32. \end{cases} \]

(b) Since \( f_Y \) is the derivative of the CDF found in part (a),

\[ f_Y(c) = \begin{cases} \frac{1}{5}c^{-4/5} & 1 \leq c \leq 32 \\ 0 & \text{else} \end{cases} \]

(c) By LOTUS, \( E[Y] = E[(1 + X)^5] = \int_0^1 (1 + u)^5 \cdot 1 du = \frac{1}{6}(1 + u)^6 \bigg|_0^1 = 2^6 - \frac{1}{6} \]

\[ = \frac{21}{2}. \] Alternatively, \( E[Y] = \int_1^{32} c \cdot \frac{1}{5}c^{-4/5} dc = \int_1^{32} \frac{1}{5}c^{1/5} dc = \frac{1}{6}c^{6/5} \bigg|_1^{32} = (\frac{5}{6})^{6} - \frac{1}{6} = \frac{21}{2}. \)