University of Illinois Spring 2008

$\mathsf{ECE\,313}$: Problem Set 13

Functions of Random Variables, Conditional Densities, Correlation and Covariance, Minimum-Mean-Square-Error Estimation

This Problem Set contains five problems

Due:	Wednesday April 30 at the beginning of class.
Reading:	Ross, Chapter 7, Sections 1-5, Chapter 8, Sections 1-4
Noncredit exercises:	Ross Chapter 7: Problems 1, 16, 26, 30 33, 34, 38;
	Theoretical Exercises: 1, 2, 17, 22, 23, 40
	Chapter 8: problems 1-9, 15.

1. (Unbelievable but true: this problem is *much* easier than it looks ...).

   (a) If $\mathcal{X} \sim \mathsf{Gaussian}(0, \sigma^2)$, use the magic formula in Example 7b, Chapter 5.7 of Ross to show that $\mathcal{X}^2 \sim \mathsf{Gamma}\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$.

   (b) Now, suppose that $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ are *independent* $\mathsf{Gaussian}(0, \sigma^2)$ random variables. Then $\mathcal{X}^2, \mathcal{Y}^2$, and $\mathcal{Z}^2$ are independent $\mathsf{Gamma}\left(\frac{1}{2}, \frac{1}{2\sigma^2}\right)$ random variables. Use the comment that immediately follows the proof of Proposition 3.1 of Ross (p. 281) to *state* what is the type of pdf of $\mathcal{W} = \mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2$, and *write down* explicitly the exact function $f_{\mathcal{W}}(\alpha)$. What is the numerical value of $f_{\mathcal{W}}(5)$ if $\sigma^2 = 4$?

   (c) Prove that $\mathsf{E}[\mathcal{W}] = 3\sigma^2$. If you actually evaluated an integral to get this answer instead of using LOTUS, shame on you!

   (d) In a physical application, $\mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ represent the velocity (measured along three perpendicular axes) of a gas molecule of mass m. Thus, $\mathcal{H} = \frac{1}{2}m\mathcal{W}$ is the kinetic energy of the particle. It is an axiom of statistical mechanics that the *average* kinetic energy is $\mathsf{E}[\mathcal{H}] = \mathsf{E}[\frac{1}{2}m\mathcal{W}] = \frac{1}{2}m\mathsf{E}[\mathcal{W}] = \frac{3}{2}m\sigma^2 = \frac{3}{2}kT$ where k is Boltzmann's constant and T is the absolute temperature of the gas in $^\circ K$. (Note that the average energy is $\frac{1}{2}kT$ per dimension.) Show that the kinetic energy $\mathcal{H}$ has the Maxwell-Boltzmann pdf:
   $f_{\mathcal{H}}(\beta) = \dfrac{2}{\sqrt{\pi}}(kT)^{-\frac{3}{2}}\sqrt{\beta}\exp\left(-\dfrac{\beta}{kT}\right)$ for $\beta \geq 0$.

   (e) $\mathcal{V} = \sqrt{\mathcal{W}} = \sqrt{\mathcal{X}^2 + \mathcal{Y}^2 + \mathcal{Z}^2}$ is the *speed* of the molecule. Show that the pdf of $\mathcal{V}$ is $f_{\mathcal{V}}(\gamma) = \dfrac{4}{\sqrt{\pi}}\left(\dfrac{m}{2kT}\right)^{\frac{3}{2}}\gamma^2\exp\left(-\dfrac{m\gamma^2}{2kT}\right)$, for $\gamma \geq 0$. cf. Theoretical Exercise 1 of Chapter 5.

   (f) What is the average speed of the molecule?

2. Let $\mathsf{E}[\mathcal{X}] = 1, \mathsf{E}[\mathcal{Y}] = 4, \mathsf{var}(\mathcal{X}) = 4, \mathsf{var}(\mathcal{Y}) = 9$, and $\rho_{\mathcal{X},\mathcal{Y}} = 0.1$.

   (a) If $\mathcal{Z} = 2(\mathcal{X} + \mathcal{Y})(\mathcal{X} - \mathcal{Y})$, what is $\mathsf{E}[\mathcal{Z}]$?

   (b) If $\mathcal{T} = 2\mathcal{X} + \mathcal{Y}$ and $\mathcal{U} = 2\mathcal{X} - \mathcal{Y}$, what is $\mathsf{cov}(\mathcal{T}, \mathcal{U})$?

   (c) Find the mean and variance of $\mathcal{W} = 3\mathcal{X} + \mathcal{Y} + 2$.

   (d) If $\mathcal{X}$ and $\mathcal{Y}$ are jointly Gaussian random variables, and $\mathcal{W}$ is as defined in part (c), what is $P\{\mathcal{W} > 0\}$?

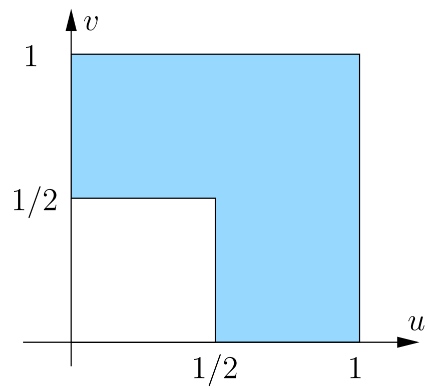
3. This problem has three independent parts. Do not apply the numbers from one part to the others.

   (a) If $\mathsf{var}(\mathcal{X} + \mathcal{Y}) = 36$ and $\mathsf{var}(\mathcal{X} - \mathcal{Y}) = 64$, what is $\mathsf{cov}(\mathcal{X}, \mathcal{Y})$? If you are also told that $\mathsf{var}(\mathcal{X}) = 3 \cdot \mathsf{var}(\mathcal{Y})$, what is $\rho_{\mathcal{X},\mathcal{Y}}$?

(b) If $\mathsf{var}(\mathcal{X}+\mathcal{Y}) = \mathsf{var}(\mathcal{X}-\mathcal{Y})$, are $\mathcal{X}$ and $\mathcal{Y}$ uncorrelated ?

(c) If $\mathsf{var}(\mathcal{X}) = \mathsf{var}(Y)$, are$\mathcal{X}$ and $\mathcal{Y}$ uncorrelated ?

4. The random point $(\mathcal{X}, \mathcal{Y})$ is uniformly distributed on the shaded region shown.



(a) Find the marginal pdf $f_{\mathcal{X}}(u)$ of the random variable $\mathcal{X}$. Find $\mathsf{E}[\mathcal{X}]$ and $\mathsf{var}(\mathcal{X})$.

(b) *Write down* the marginal pdf $f_{\mathcal{Y}}(v)$ of the random variable $\mathcal{Y}$, and its mean and variance, from your answer to part (a).

(c) Find $f_{\mathcal{Y}|\mathcal{X}}(v|\alpha)$, the conditional pdf of $\mathcal{Y}$ given that $\mathcal{X} = \alpha$, where $0 < \alpha < 1/2$. *Write down* the conditional mean and conditional variance of $\mathcal{Y}$ given $\mathcal{X} = \alpha$.
Find $f_{\mathcal{Y}|\mathcal{X}}(u|\alpha)$, the conditional pdf of $\mathcal{Y}$ given that $\mathcal{X} = \alpha$, where $1/2 < \alpha < 1$. *Write down* the conditional mean and conditional variance of $\mathcal{Y}$ given $\mathcal{X} = \alpha$.

(d) Now, apply the theorem of total probability to compute $f_{\mathcal{Y}}(v)$, the *unconditional* pdf of $\mathcal{Y}$ from $f_{\mathcal{Y}|\mathcal{X}}(v|\alpha)$. Do you get the same answer as in part (b)? Why not?

(e) *Given* that the value of $\mathcal{X}$ is u, the (conditional) minimum-mean-square-error estimate of $\mathcal{Y}$ is $\mathsf{E}[\mathcal{Y}|\mathcal{X} = u]$, the conditional mean of $\mathcal{Y}$, and the (conditional minimum) mean-square error achieved is $\mathsf{var}(\mathcal{Y}|\mathcal{X} = u)$. Use your answers to part (c) to sketch graphs of $\mathsf{E}[\mathcal{Y}|\mathcal{X} = u]$ and $\mathsf{var}(\mathcal{Y}|\mathcal{X} = u)$ as functions of u for $0 < u < 1$.

(f) Since $\mathsf{var}(\mathcal{Y}|\mathcal{X} = u)$ depends on the value of $\mathcal{X}$, it is a *function* of $\mathcal{X}$. Use LOTUS to compute $\mathsf{E}[\mathsf{var}(\mathcal{Y}|\mathcal{X} = u)]$, the expected value of this function. This is the (unconditional) mean-square error that we achieve when we estimate $\mathcal{Y}$ as $\mathsf{E}[\mathcal{Y}|\mathcal{X} = u]$, and *no* other estimate can have smaller mean-square error than this.

(g) Compute $\mathsf{E}[\mathcal{X}\mathcal{Y}]$ and hence determine $\mathsf{cov}(\mathcal{X}, \mathcal{Y})$ and $\rho_{\mathcal{X},\mathcal{Y}}$.

(h) The minimum-mean-square-error *linear* estimate of $\mathcal{Y}$ given that the value of $\mathcal{X}$ is u is $\hat{\mathcal{Y}} = \mathsf{E}[\mathcal{X}] + \rho_{\mathcal{X},\mathcal{Y}}\sqrt{\mathsf{var}(\mathcal{Y})/\mathsf{var}(\mathcal{X})}\big(u - \mathsf{E}[\mathcal{X}]\big)$, and the (unconditional) mean-square error of this linear estimate is $\mathsf{var}(\mathcal{Y})(1-\rho^2_{\mathcal{X},\mathcal{Y}})$. Sketch $\hat{\mathcal{Y}}$ as a function of u on the same graph that you used in part (e) and compare to the nonlinear (optimum) estimate $\mathsf{E}[\mathcal{Y}|\mathcal{X} = u]$. Is $\mathsf{E}[\mathsf{var}(\mathcal{Y}|\mathcal{X} = u)] \leq \mathsf{var}(\mathcal{Y})(1 - \rho^2_{\mathcal{X},\mathcal{Y}})$ as it should be?

5. Suppose that $\mathcal{X}$ and $\mathcal{Y}$ are zero-mean jointly Gaussian random variables with variances σ_1^2 and σ_2^2 respectively and correlation coefficient ρ.

(a) Find the means and variances of the random variables $\mathcal{Z} = \mathcal{X}\cos\theta + \mathcal{Y}\sin\theta$ and $\mathcal{W} = \mathcal{Y}\cos\theta - \mathcal{X}\sin\theta$.

(b) What is $\mathsf{cov}(\mathcal{Z}, \mathcal{W})$?

(c) Find an angle θ such that $\mathcal{Z}$ and $\mathcal{W}$ are *independent* Gaussian random variables. You may express your answer as a trigonometric function involving σ_1^2, σ_2^2 , and ρ. In particular, what is the value of θ if $\sigma_1 = \sigma_2$?