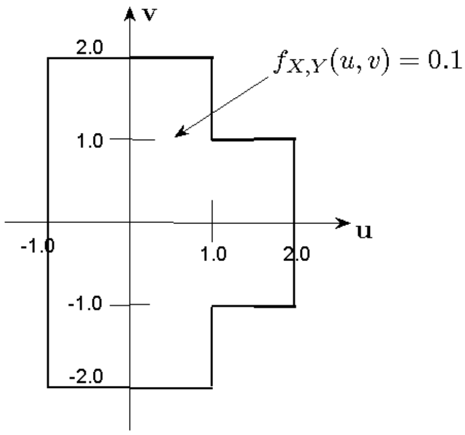
University of Illinois Spring 2008

ECE 313: Problem Set 12
Many Random Variables

This Problem Set contains five problems

Due: Wednesday April 23 at the beginning of class.
Reading:
Noncredit Exercises:

1. Random variables X and Y have the joint PDF shown below:



   (a) Prepare neat, fully labeled sketches of $f_{X|Y}(u|v)$.

   (b) Find $\mathsf{E}[X|Y = v]$ and $\mathsf{Var}[X|Y = v]$.

   (c) Find $\mathsf{E}[X]$.

2. Continuous random variables X and Y have the joint PDF

$$f_{X,Y}(u,v) = \begin{cases} \frac{u}{8}, & \text{for } 1 \le u \le 3 \text{ and } 0 \le v \le 2 \\ 0, & \text{otherwise} \end{cases}$$

   Let R be defined by $R = Y/X$ and let D be the event $R \le 1/3$.

   (a) Are X and Y independent? Are they independent conditioned on event D? Given convincing arguments.

   (b) Determine the numerical value of $P(D)$.

   (c) Obtain and carefully sketch the PDF's $f_X(u)$ and $f_{X|D}(u|D)$.

   (d) Determine the conditional PDF for R given D has occurred.

3. X_1 and X_2 are independent random variables. X_1 is uniformly distributed over $[-1, 1]$ and X_2 is exponentially distributed with parameter $\lambda = 1$. Find the pdfs of the RVs $W = X_1 + X_2$ and $Z = X_1/X_2$.

4. $\mathcal{X}$ and $\mathcal{Y}$ denote *independent* standard Gaussian random variables.

   (a) What is the joint pdf $f_{\mathcal{X},\mathcal{Y}}(u,v)$ of $\mathcal{X}$ and $\mathcal{Y}$?

   (b) Sketch the u-v plane and indicate on it the region over which you need to integrate the joint pdf in order to find $P\{\mathcal{X}^2+\mathcal{Y}^2 > 2\alpha^2\}$. Compute $P\{\mathcal{X}^2+\mathcal{Y}^2 > 2\alpha^2\}$.

   (c) Let $\mathcal{Z} = \mathcal{X}^2+\mathcal{Y}^2$. What is the pdf of $\mathcal{Z}$?

   (d) Express $P\{|\mathcal{X}| > \alpha\}$ in terms of the complementary unit Gaussian CDF function $Q(x)$, and use this to write $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\}$ in terms of $Q(x)$. (Remember commas mean intersections).

   (e) On your sketch of part (b), show the region over which you must integrate the joint pdf to find $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\}$. Use your sketch to prove the following result: $P\{|\mathcal{X}| > \alpha, |\mathcal{Y}| > \alpha\} < P\{\mathcal{X}^2+\mathcal{Y}^2 > 2\alpha^2\}$ for $\alpha > 0$.

   (f) Show that inequality of part (e) implies that $Q(x) < \frac{1}{2}\exp(-x^2/2)$ for $x > 0$.

   (g) On your sketch of parts (b) and (d), show the region over which you must integrate to find $P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\}$, and prove that

   $$P\{\mathcal{X}^2+\mathcal{Y}^2 \leq \alpha^2\} < P\{|\mathcal{X}| < \alpha, |\mathcal{Y}| < \alpha\} < P\{\mathcal{X}^2+\mathcal{Y}^2 < 2\alpha^2\}.$$

   Use these inequalities to deduce the *lower* bound $Q(x) > \frac{1}{4}\exp(-x^2)$ for $x > 0$. Note that at $x = 0$, equality holds in the upper bound of part (f) but not in this lower bound .

5. Consider random variables X and Y that have the following joint PDF

   $$f_{X,Y}(u,v) = c\sqrt{u^2+v^2}$$

   where u, v belong to the unit circle centered at the origin.

   (a) Find c.

   (b) Using LOTUS, compute $\mathsf{E}[f(X,Y)]$.

   (c) Calculate $P\{f(X,Y) \geq \frac{1}{2}\}$.