University of Illinois Spring 2008

$\mathsf{ECE\,313}$: Problem Set 11
Many Random Variables

This Problem Set contains six problems

Due:	Wednesday April 16 at the beginning of class.
Reading:	Ross Chapter 6
Noncredit Exercises:	Chapter 6: Problems 1, 9-15, 20-23
Reminder:	Hour Exam II on Monday April 14, 7:00 p.m. – 8:00 p.m Room 165 Everitt Laboratory

1. The joint pmf $p_{\mathcal{X},\mathcal{Y}}(u,v)$ of $\mathcal{X}$ and $\mathcal{Y}$ is shown in the table below.

$u \rightarrow$ $v \downarrow$	0	1	3	5
4	0	1/12	1/6	1/12
3	1/6	1/12	0	1/12
-1	1/12	1/6	1/12	0

   (a) Find the marginal pmfs $p_{\mathcal{X}}(u)$ and $p_{\mathcal{Y}}(v)$.

   (b) Are $\mathcal{X}$ and $\mathcal{Y}$ independent random variables?

   (c) Find $P\{\mathcal{X} \leq \mathcal{Y}\}$ and $P\{\mathcal{X} + \mathcal{Y} \leq 4\}$.

   (d) Find $p_{\mathcal{X}|\mathcal{Y}}(u|3)$, $\mathsf{E}[\mathcal{X}|\mathcal{Y} = 3]$ and $\mathsf{var}(\mathcal{X}|\mathcal{Y} = 3)$.

2. Packets arriving at a router are addressed either to Server A or to Server B. Each packet address (A or B) may be regarded as an independent trial of an experiment whose outcomes are A and B with probabilities p and $q = 1 - p$ respectively. The router sends each packet to the appropriate server.

   We model the packet arrivals as a Poisson process with arrival rate λ. Let $\mathcal{X} = N(0, T]$ denote the number of packets arriving at the router during $(0, T]$ and $\mathcal{Y} = N_A(0, T]$ the number of packets that are routed to Server A. Then, all the above implies that *given* that $\mathcal{X} = n$, the *conditional* pmf $p_{\mathcal{Y}|\mathcal{X}}(m|n)$ of $\mathcal{Y}$ is binomial with parameters (n, p). Of course, $\mathcal{X}$ is a Poisson random variable with parameter λT.

   (a) Show that the *unconditional* pmf of $\mathcal{Y}$ is Poisson with parameter $(\lambda p)T$.

   (b) Explain briefly why the *unconditional* pmf of $\mathcal{Z} = \mathcal{X} - \mathcal{Y}$ is Poisson with parameter $(\lambda q)T$. Note that $\mathcal{Z} = N_B(0, T]$ counts the number of packets that are addressed to Server B.

   The packet streams routed to the two servers are in fact Poisson processes with arrival rates λp and λq respectively. The division of a Poisson stream into two Poisson substreams is called *Poisson splitting.* Poisson splitting also occurs if the packets are jobs and it is the router that chooses to randomly allot the jobs to Server A or B with probabilities p and q. However, Poisson splitting *does not* occur if the incoming jobs are simply routed alternately to Servers A and B: the probability that Server A receives k jobs in $(0, T]$ is $P\{\mathcal{X} = 2k-1\} + P\{\mathcal{X} = 2k\}$ which does not match up with a Poisson probability mass function.

(c) What is the conditional pmf $p_{\mathcal{X}|\mathcal{Y}}(n|m)$ of $\mathcal{X}$ given that $\mathcal{Y}$ was observed to have value m?

(d) What is $\mathsf{E}[\mathcal{X}|\mathcal{Y}=m]$, the conditional expectation of $\mathcal{X}$ given that $\mathcal{Y}$ was observed to have value m?

3. The jointly continuous random variables $\mathcal{X}$ and $\mathcal{Y}$ have joint pdf

$$f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} 0.5, & 0 \le u < 1, 0 \le v < 1, \text{ and } 0 \le u+v < 1, \\ 1.5, & 0 \le u < 1, 0 \le v < 1, \text{ and } 1 \le u+v < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Find the marginal pdf of $\mathcal{X}$.

(b) Find $P\{\mathcal{X}+\mathcal{Y} \le 3/2\}$ and $P\{\mathcal{X}^2+\mathcal{Y}^2 \ge 1\}$.

4. Ross, Problem 8, page 313.

5. The jointly continuous random variables $\mathcal{X}$ and $\mathcal{Y}$ have joint pdf

$$f_{\mathcal{X},\mathcal{Y}}(u,v) = \begin{cases} 2\exp(-u-v), & 0 < u < v < \infty, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Sketch the u-v plane and indicate on it the region over which $f_{\mathcal{X},\mathcal{Y}}(u,v)$ is nonzero.

(b) Find the marginal pdfs of $\mathcal{X}$ and $\mathcal{Y}$.

(c) Are the random variables $\mathcal{X}$ and $\mathcal{Y}$ independent ?

(d) Find $P\{\mathcal{Y} > 3\mathcal{X}\}$.

(e) For $\alpha > 0$, find $P\{\mathcal{X}+\mathcal{Y} \le \alpha\}$.

(f) Use the result in part (e) to determine the pdf of the random variable $\mathcal{Z} = \mathcal{X}+\mathcal{Y}$.

6. We return to the "random chord" of Problem 4 of Problem Set 10. Yet another way of defining a "random chord" is to choose the midpoint of the chord to be anywhere inside the circle of radius 1 with equal probability. The chord is, of course, perpendicular to the diameter that passes through the chosen point. Thus, let the random point $(\mathcal{X},\mathcal{Y})$ be *uniformly distributed* on the interior of the circle of unit radius centered at the origin (this region is called the unit disc — nomenclature that might be familiar to DSPists).

(a) Find the probability that the length $\mathcal{L}$ of the random chord is greater than the side of the equilateral triangle inscribed in the circle.

(b) Express $\mathcal{L}$ as a function of the random variable $(\mathcal{X},\mathcal{Y})$ and find the probability density function for $\mathcal{L}$.

(c) Find the average length of the chord, i.e. find $\mathsf{E}[\mathcal{L}]$.