University of Illinois Spring 2008

ECE 313: Problem Set 10

Functions of Continuous Random Variables; and Hypothesis Testing

This Problem Set contains 6 problems

Due:	Wednesday April 9 at the beginning of class.
Reading:	Ross, Chapter 5; Powerpoint Lectures 26, 27, and 29
Noncredit Exercises:	Ross, Chapter 5, Problems 10-19; 21, 22, 24, 31-41

1. The current I through a semiconductor diode is related to the voltage V across the diode as $I = I_0(\exp(V)-1)$ where I_0 is the magnitude of the reverse current. Suppose that the voltage across the diode is modeled as a continuous random variable $\mathcal{V}$ with pdf
$$f_{\mathcal{V}}(u) = 0.5\exp(-|u|), -\infty < u < \infty.$$
Then, the current $\mathcal{I}$ is also a continuous random variable.
   (a) What values can $\mathcal{I}$ take on?
   (b) Find the CDF of $\mathcal{I}$.
   (c) Find the pdf of $\mathcal{I}$.

2. $\mathcal{X}$ is uniformly distributed on $[-1, +1]$.
   (a) Find the pdf of $\mathcal{Y} = \mathcal{X}^2$.
   (b) Find the pdf of $\mathcal{Z} = g(\mathcal{X})$ where $g(u) = \begin{cases} u^2, & u \geq 0 \\ -u^2, & u < 0 \end{cases}$.

3. ["Give me an A! Give me a D! Give me a converter! What have we got? An A/D converter! Go Team!"] A signal $\mathcal{X}$ is modeled as a unit Gaussian random variable. For some applications, however, only the quantized value $\mathcal{Y}$ (where $\mathcal{Y} = \alpha$ if $\mathcal{X} > 0$ and $\mathcal{Y} = -\alpha$ if $\mathcal{X} \leq 0$) is used. Note that $\mathcal{Y}$ is a *discrete* random variable.
   (a) What is the pmf of $\mathcal{Y}$?
   (b) The *squared error* in representing $\mathcal{X}$ by $\mathcal{Y}$ is $\mathcal{Z} = \begin{cases} (\mathcal{X}-\alpha)^2, & \text{if } \mathcal{X} > 0, \\ (\mathcal{X}+\alpha)^2, & \text{if } \mathcal{X} \leq 0, \end{cases}$ and varies as different trials of the experiment produce different values of $\mathcal{X}$. We would like to choose the value of α so as to minimize the *mean* squared error $\mathsf{E}[\mathcal{Z}]$. Use LOTUS to easily calculate $\mathsf{E}[\mathcal{Z}]$ (the answer will be a function of α), and then find the value of α that minimizes $\mathsf{E}[\mathcal{Z}]$.
   (c) We now get more ambitious and use a 3-bit A/D converter which first quantizes $\mathcal{X}$ to the nearest integer $\mathcal{W}$ in the range -3 to $+3$. Thus, $\mathcal{W} = 3$ if $\mathcal{X} \geq 2.5$, $\mathcal{W} = 2$ if $1.5 \leq \mathcal{X} < 2.5$, $\mathcal{W} = 1$ if $0.5 \leq \mathcal{X} < 1.5$, $\cdots$, $\mathcal{W} = -3$ if $\mathcal{X} < -2.5$. Note that $\mathcal{W}$ is also a discrete random variable. Find the pmf of $\mathcal{W}$.
   (d) The output of the A/D converter is a 3-bit 2's complement representation of $\mathcal{W}$. Suppose that the output is $(\mathcal{Z}_2, \mathcal{Z}_1, \mathcal{Z}_0)$. What is the pmf of $\mathcal{Z}_2$? the pmf of $\mathcal{Z}_1$? the pmf of $\mathcal{Z}_0$? Note that $(1,0,0)$ which represents -4 is not one of the possible outputs from this A/D converter.

4. [Read Example 3d (pp. 217-218) in Chapter 5 of Ross first] Let the straight line segment ACB be a diameter of a circle of unit radius and center C. Consider an *arc* AD of the circle where the length $\mathcal{X}$ of the arc (measured clockwise around the circle) is a random variable uniformly distributed on $[0, 2\pi)$. Now consider the *random chord* AD whose length we denote by $\mathcal{L}$.

   (a) Find the probability that $\mathcal{L}$ is greater than the side of the equilateral triangle inscribed in the circle.

   (b) Express $\mathcal{L}$ as a function of the random variable $\mathcal{X}$, and find the pdf for $\mathcal{L}$.

5. If hypothesis H_0 is true, the pdf of $\mathcal{X}$ is exponential with parameter 5 while if hypothesis H_1 is true, the pdf of $\mathcal{X}$ is exponential with parameter 10.

   (a) Sketch the two pdfs.

   (b) State the *maximum-likelihood* decision rule in terms of a threshold test on the *observed value* u of the random variable $\mathcal{X}$ instead of a test that involves comparing the likelihood ratio $\Lambda(u) = f_1(u)/f_0(u)$ to 1.

   (c) What are the probabilities of false-alarm and missed detection for the maximum-likelihood decision rule of part (b)?

   (d) The Bayesian (minimum probability of error) decision rule compares $\Lambda(u)$ to π_0/π_1. Show that this decision rule also can be stated in terms of a threshold test on the observed value u of the random variable $\mathcal{X}$.

   (e) If $\pi_0 = 1/3$, what is the *average* probability of error of the Bayesian decision rule?

   (f) What is the average error probability of a decision rule that always decides H_1 is the true hypothesis, regardless of the value taken on by $\mathcal{X}$?

   (g) Show that if $\pi_0 > 2/3$, the Bayesian decision rule always decides that H_0 is the true hypothesis regardless of the value taken on by $\mathcal{X}$. What is the average probability of error for the maximum-likelihood rule when $\pi_0 > 2/3$?

6. Consider the following hypothesis testing problem:

$$
\begin{aligned}
\mathsf{H}_0 &: X \sim \mathcal{N}(0, \sigma_0^2) \\
\mathsf{H}_1 &: X \sim \mathcal{N}(0, \sigma_1^2),
\end{aligned}
$$

   where $\sigma_0^2 < \sigma_1^2$.

   (a) Sketch the two pdfs.

   (b) Show that both the ML decision rule and the Bayes decision rule simplify to comparing $|X|$ to a threshold. Specify the threshold in both cases.
   (*Hint: consider the log likelihood ratio. The threshold is a function of* σ_0^2 *and* σ_1^2*, (and of* π_0 *and* π_1*)).*

   (c) Calculate the false alarm and missed detection probabilities for the ML decision rule for the case $\sigma_0^2 = 1, \sigma_1^2 = 4$.