University of Illinois Spring 2008

$\textsf{ECE 313}$: Problem Set 9
Continuous Random Variables; Poisson Processes; and Gaussian Random Variables

This Problem Set contains seven problems

Due:	Wednesday April 2 at the beginning of class.
Reading:	Ross Ch. 5
Noncredit Exercises:	**Chapter 5:** Problems 10-19; 21, 22, 24, 31-41

1. $\mathcal{X}$ is uniformly distributed on $[-1, +1]$.
   (a) If $\mathcal{Y} = \mathcal{X}^2$, what are the mean and variance of $\mathcal{Y}$?
   (b) If $\mathcal{Z} = g(\mathcal{X})$ where $g(u) = \begin{cases} u^2, & u \geq 0 \\ -u^2, & u < 0 \end{cases}$ use LOTUS to find $\mathsf{E}[Z]$
   (c) On a completely unrelated LOTUSian question, if $\mathcal{U}$ is a geometric random variable with parameter $\frac{1}{2}$, and $\mathcal{V} = \sin(\pi\mathcal{U}/2)$, what is the value of $\mathsf{E}[\mathcal{V}]$?

2. Consider a sphere whose radius is a random variable $\mathcal{R}$ with pdf $f_{\mathcal{R}}(u) = 2u,\ 0 < u < 1$, and 0 otherwise.
   (a) What is the average radius of the sphere? What is the average volume? What is the average surface area? If a sphere of average radius is called an *average sphere*, then does an average sphere have the average volume? Does it have the average surface area?
   (b) Show that $\mathsf{E}[\mathcal{R}] > \mathsf{E}[\mathcal{R}^2] > \mathsf{E}[\mathcal{R}^3]$ for *any* pdf for $\mathcal{R}$ that is nonzero only on the unit interval $(0, 1)$.

3. Consider a Poisson process with arrival rate λ.
   (a) What is the mean number of arrivals in the interval $(0, 4]$? That is, what is $\mathsf{E}[N(0, 4]]$?
   (b) What is $P[\{N(0, 3] = 3\} \cap \{N(2, 6] = 0\}]$?
   (c) If we observe that there were 5 arrivals in $(0, 6]$, what is the maximum-likelihood estimate of the arrival rate λ?
   (d) Now suppose that $\lambda = \ln 2$. What is the probability that at least one arrival occurs in $(0, t]$?

4. Todd breaks the chalk piece with which he is writing on the blackboard at random times that can be modeled as arrivals in a Poisson process with arrival rate $\lambda = 0.1$ per minute.
   (a) What is the expected length of time between two successive chalk breaks?
   (b) What is the average number of times that Todd breaks the chalk during a 50 minute lecture?
   (c) *Given* that Todd broke 6 chalk pieces in 50 minutes, what is the average number of pieces he broke in the first 25 minutes?

5. Let $\mathcal{X}$ denote a Gaussian random variable with mean -10 and variance $\sigma^2 = 4$. You have at your disposal two calculators: one can calculate $\Phi(x)$ for $x \geq 0$, and the other can calculate $Q(x)$ for $x \geq 0$. Both have the usual assortment of basic arithmetic functions. Write down expressions for calculating each of the following probabilities with each of the calculators. Remember that the argument of Φ or Q must be ≥ 0 in all cases.
(**a**) $P\{\mathcal{X} < 0\}$. (**b**) $P\{-10 < \mathcal{X} < 5\}$. (**c**) $P\{\,|\mathcal{X}| \geq 5\}$. (**d**) $P\{\mathcal{X}^2 - 3\mathcal{X} + 2 > 0\}$.

6. The width of a metal trace on a circuit board is modelled as a Gaussian random variable with mean $\mu = 0.9$ microns and standard deviation $\sigma = 0.003$ microns.

   (a) Traces that fail to meet the requirement that the width be in the range 0.9 ± 0.005 microns are said to be defective. What percentage of traces are defective?

   (b) A new manufacturing process that produces smaller variations in trace widths is to be designed so as to have no more than 1 defective trace in 100. What is the maximum value of σ for the new process if the new process achieves the goal?

7. A signal $x(t) = \exp(-\pi t^2), -\infty < t < \infty$, is the input to an ideal low-pass filter whose transfer function is $H(f) = \text{rect}(f/2)$. Let $y(t)$ denote the output of the filter. Find the *numerical* value of $y(0)$.

   [Hint: $X(f) = \displaystyle\int_{-\infty}^{\infty} x(t)\exp(-j2\pi f t)\,dt = \exp(-\pi f^2), -\infty < f < \infty$, is the Fourier transform of $x(t)$ with respect to the frequency variable f as opposed to the *radian* frequency variable ω that you used in ECE 210. The inverse Fourier transform is defined as $x(t) = \displaystyle\int_{-\infty}^{\infty} X(f)\exp(j2\pi f t)\,df$. Similarly, $H(f)$ is the Fourier transform with respect to f of the filter impulse response $h(t)$. Fortunately, it is still true that $Y(f) = H(f)X(f)$ and so your time in ECE 210 was not totally wasted...]