University of Illinois Spring 2008

ECE 313: Problem Set 8
Continuous Random Variables

This Problem Set contains 6 problems

Due:	Wednesday March 26 at the beginning of class.
Reading:	Chapter 5
Noncredit Exercises:	Chapter 5, Problems 1-6, 15-19, 23-25, 32-34; Theoretical Exercises 1, 8

1. Which of the following functions $F(u)$ are valid CDFs? For those that are valid CDFs, compute the probability that the absolute value of the random variable exceeds 0.5.

   (a) $F(u) = \begin{cases} 0 & u < 0, \\ u^2, & 0 \le u < 1, \\ 1, & u \ge 1. \end{cases}$ (b) $F(u) = \begin{cases} 0 & u < 1, \\ 2u - u^2, & 1 \le u \le 2, \\ 1, & u > 2. \end{cases}$

   (c) $F(u) = \begin{cases} \frac{1}{2}\exp(2u) & u \le 0, \\ 1 - \frac{1}{4}\exp(-3u), & u > 0, \end{cases}$ (d) $F(u) = \begin{cases} \frac{1}{2}\exp(2u) & u < 0, \\ 1 - \frac{1}{4}\exp(-3u), & u \ge 0, \end{cases}$

2. Let X be a random variable with CDF

$$F_X(b) = \begin{cases} 0, & b < 1 \\ (1/3)b, & 1 \le b < 2 \\ 1, & b \ge 2 \end{cases}$$

   (a) Sketch this CDF and determine whether X is a discrete, continuous, or mixed random variable.
   (b) Find $\mathrm{E}[X]$.
   (c) Evaluate the probability that $|X - 1| < 1$.
   (d) Evaluate the conditional probability of $|X - 1| < 1$ given that $1 < X \le 2$, that is $P(|X - 1| < 1 | 1 < X \le 2)$.

3. Let X have density $f_X(u) = \frac{1}{2}e^{-|u|}$.

   (a) Sketch the graph of f.
   (b) Find $P(|X| < 4)$.
   (c) Find $P(X^2 + X > 0)$.

4. For each density $f_X(u)$, find the corresponding distribution function $F_X(u)$.

   (a) $f_X(u) = \frac{100}{u^2}$ for $u \ge 100$
   (b) $f_X(u) = 1 - |u|$ for $|u| < 1$
   (c) $f_X(u) = \begin{cases} 4 & \text{if } -0.1 \le u < 0.1 \\ \frac{1}{2} & \text{if } 0.1 \le u \le 0.5 \end{cases}$

5. The amount of bread (in hundreds of pounds) that a bakery sells in a day is a random variable X with probability density function

$$f_X(u) = \begin{cases} cu, & 0 \le u < 3 \\ c(6-u), & 3 \le u < 6 \\ 0, & \text{otherwise} \end{cases}$$

   (a) Find the value of the constant c.

   (b) Compute the CDF $F_X(u)$ of X.

   (c) What is the probability that the number of pounds of bread sold in a single day will be (i) more than 300 pounds? (ii) between 150 and 900 pounds?

   (d) If A and B are the events in (i) and (ii), respectively, are these events independent?

6. Let X be a random variable that denotes the weekly demand (measured in thousands of gallons) for gasoline at a particular gas station. The pdf for X is given by

$$f_X(u) = \begin{cases} u, & 0 \le u \le 1 \\ 2-u, & 1 \le u \le 2 \end{cases}$$

   Let C denote the capacity (in thousands of gallons) of the station's storage tank, which is refilled weekly.

   The owner of the gas station is very interested in the random variable Y, equal to the amount (measured in thousands of gallons) of gasoline sold in a given week. Note that the amount of gasoline sold cannot exceed the tanks capacity, i.e., $Y \le C$.

   Assume that the gross profit for each gallon sold is \$ 0.64.

   (a) Sketch the density f_X on [0,2], and verify that it is indeed a probability density function.

   (b) Suppose that $C = 1$ and it is observed that $X = 1.105$. Can the gas station satisfy the demand?

   (c) Plot the probability that demand is satisfied, as a function of C, using a computer program such as Matlab. What is the probability that demand is satisfied when $C = 1$?

   (d) What is the minimum value for C to ensure that the probability that the demand exceeds the supply is no larger than 10^{-1}?

   (e) Plot the expected value of the weekly gross profit, as a function of C.

   (f) Suppose that the owner pays \$ $10C$ as a weekly rent on a tank of capacity $1000C$ gallons. Plot the expected value of the weekly net profit as a function of C. What value of C will maximize profit?